On The Vertex Choice Number And Chromatic Number Of Graphs

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Abstract: Let G = (V; E) be a connected, simple graph of order n and size m and let $V(G) = \{1, 2, ..., n\}$. A graph G = (V; E) is said to be vertex (n, k)-choosable, if there exists a collection of subsets of the vertex set, $\{S_k(v): v \in V(G)\}$ of cardinality k, such that $S_k(u) \cap S_k(v) = \emptyset$ for all $uv \in E(G)$. The chromatic number $\chi(G)$ of a graph G is the minimum number of colors needed to color the vertices of G, so that no two adjacent vertices share the same color. This paper initiates a study on vertex choice number and chromatic number of a given graph G, and find the relation between them.

Keywords: Chromatic number, $(a{:}b)$-choosability, vertex choice number, χ-choosability.

1. INTRODUCTION

Throughout this article, unless otherwise mentioned, by a graph we mean a connected, simple graph and any terms which are not mentioned here, the reader may refer to [1]. Graph coloring is considered as the special case of graph labeling, in which the labels are called colors and it is assigned to the elements of a graph subject to certain conditions. Vertex coloring is a way of coloring the vertices of the graph where no two adjacent vertices receive the same color. The chromatic number $\chi(G)$ of a graph G is the minimum number of colors needed to color the vertices of G, so that no two adjacent vertices share the same color. A list assignment $\mathcal{L}$ (or a list coloring) of a graph G is a mapping that assigns to every vertex v of, a finite list $\mathcal{L}(v)$ of colors. Invoking the concept of list-assignments of graphs, the concept of $(a{:}b)$-choosability was defined and studied by in [6] as: a graph $G = (V, E)$ is $(a{:}b)$-choosable, if for every family of sets $\{S(v): v \in V(G)\}$ of cardinality a, there exist subsets $C(v) \subset S(v)$, where $|C(v)| = b$ for every $v \in V$, and $C(u) \cap C(v) = \emptyset$, whenever $u, v \in V$ are adjacent. Analogous to the terminology of $(a{:}b)$-choosable graph, the vertex set oriented $(a{:}b)$-choosable graph is defined in [5] as follows: A graph $G = (V, E)$ is said to be vertex $(n{:}k)$-choosable, if there exists a collection of subsets of the vertex set of G, say, $\{S_k(v): v \in V(G)\}$ of cardinality k, such that $S_k(u) \cap S_k(v) = \emptyset$ for all $uv \in E(G)$. The maximum value of such k is called the vertex choice number of G and is denoted by $\mathcal{V}_{ch}(G)$. This paper initiates a study on the vertex choice number and chromatic number of a given graph G, and try to find the relation between them. For only certain graphs the vertex choice number and the chromatic number will coincide. It is interesting to see that for all classes of graphs the vertex choice number depends on its chromatic number. Hence finding the relation between these two graph parameters and finding those graphs for which, $\mathcal{V}_{ch}(G) = \chi(G)$ is interesting.

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2 DISCUSSION AND NEW RESULTS

Using the terminology of chromatic number and vertex choice number of graphs, the notion of the χ-vertex choosable graph is defined as follows:

Definition 2.1 (χ -Vertex Choosable Graphs) : A graph G is said to be a χ -vertex choosable graph , if the vertex choice number of G is equal to its chromatic number, that is when $\mathcal{V}_{ch}(G) = \chi(G)$.

We observe that only for certain graphs the vertex choice number and the chromatic number will coincide.

Examples 2.2

- The chromatic number of an even cycle C_n is always 2. The even cycle for which 2 as the vertex choice number is C_4. Hence C_4 is χ - vertex choosable.
- The chromatic number of an odd cycle C_n is 3. The odd cycles for which 3 as the vertex choice number are C_7. Therefore C_7 is χ -vertex choosable.
- The chromatic number of a path P_n is always 2. For, P_4 and P_5 the vertex choice number is also 2. Hence P_4 and P_5 are χ -vertex choosable paths.

It is interesting to see that the vertex choice number of a graph depends on its chromatic number. In the following section we are trying to find the relation between these two graph parameters for certain classes of graphs.

Lemma 2.3. For a complete graph K_n, $((\mathcal{V}_{ch}(K_n))(\chi(K_n)) = n$.
Proof. In a complete graph K_n any two vertices are mutually adjacent. The chromatic number of K_n is n and there should be atleast n disjoint color classes. If a k -element subset of V is assigned to the first color class, then that set cannot be assigned to any other color class or any other vertex. Hence, we have to get n number of disjoint k-element subsets of n to be assigned to the vertices of. That is the vertex choice number will be $\left\lfloor \frac{n}{\chi(K_n)} \right\rfloor$, which implies $((\mathcal{V}_{ch}(K_n))(\chi(K_n)) = n$.

Lemma 2.4. Let C_n be an even cycle. Then, $(\mathcal{V}_{ch}(C_n))(\chi(C_n)) = n$.
Proof. Let C_n be an even cycle with the vertex set $\{1,2,3,...,n\}$. Starting from the first vertex all the alternating vertices will form a color class in C_n. That is, by using two disjoint colors