IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 15, Issue 2 Ser. II (Mar – Apr 2019), PP 24-31
www.iosrjournals.org

Circulant Graphs without Cayley Isomorphism Property with $m = 3$

V. Vilfred[1] and P. Wilson[2]
Department of Mathematics
[1]Central University of Kerala, Tejaswini Hills Periye – 671 316, Kasaragod, Kerala, India.
[2]S.T. Hindu College, Nagercoil – 629 002, Kanyakumari District, Tamil Nadu, India.
Corresponding Author: V. Vilfred

Abstract: *A circulant graph $C_n(R)$ is said to have the Cayley Isomorphism (CI) property if whenever $C_n(S)$ is isomorphic to $C_n(R)$, there is some $a \in Z_n^*$ for which S = aR. In this paper, we prove that $C_{27n}(R)$, $C_{27n}(S)$ and $C_{27n}(T)$ are isomorphic circulant graphs without CI-property where R = {1, 9n-1, 9n+1, $3p_1$, $3p_2$, . . . , $3p_{k-2}$}, S = {3n+1, 6n-1, 12n+1, $3p_1$, $3p_2$, . . . , $3p_{k-2}$}, T = {3n-1, 6n+1,12n-1, $3p_1$, $3p_2$, . . . ,$3p_{k-2}$}, k ≥ 3, gcd(p_1, p_2,..., p_{k-2}) = 1 and $n, p_1, p_2, ..., p_{k-2} \in \mathbb{N}$ and also obtain new abelian groups from these isomorphic circulant graphs.*

AMS Subject Classification: ***05C60, 05C25.***
Keywords: *Adam's isomorphism or Type-1 isomorphism, Type-2 isomorphism, Cayley Isomorphism (CI) property, symmetric equidistance condition, Type-1 group on $C_n(R)$, Type-2 group on $C_n(R)$ w.r.t. m.*

Date of Submission: 27-03-2019 Date of acceptance: 11-04-2019

I. Introduction

Circulant graphs have been investigated by many authors [1]-[16]. An excellent account can be found in the book by Davis [3] and in [6]. A circulant graph $C_n(R)$ is said to have the *Cayley Isomorphism (CI) property* if whenever $C_n(S)$ is isomorphic to $C_n(R)$ there is some $a \in \mathbb{Z}_n$ for which $S = aR$. Finding circulant graphs without CI-property is difficult. Type-2 isomorphism, a new type of isomorphism of circulant graphs, other than already known Adam's isomorphism, was defined and studied in [10,13]. Type-2 isomorphic circulant graphs have the property that they are isomorphic circulant graphs without CI-property.

Families of isomorphic circulant graphs of Type-2, each circulant graph of a family with $m_j = gcd(n,r_j)$ number of copies of a circulant subgraph for m_j = 2, 5 or 7 are obtained in [14]-[16]. In this paper, we prove that for $n \in \mathbb{N}$, $k \geq 3$, $R = \{1, 9n\text{-}1, 9n+1, 3p_1, 3p_2, . . . , 3p_{k-2}\}$, $S = \{3n+1, 6n\text{-}1, 12n+1, 3p_1, 3p_2, . . . , 3p_{k-2}\}$ and $T = \{3n\text{-}1, 6n+1, 12n\text{-}1, 3p_1, 3p_2, ..., 3p_{k-2}\}$, circulant graphs $C_{27n}(R)$, $C_{27n}(S)$ and $C_{27n}(T)$ are Type-2 isomorphic with $m_i = 3$ where $gcd(p_1,p_2,...,p_{k-2}) = 1$ and $p_1, p_2, ..., p_{k-2} \in \mathbb{N}$ and obtain abelian groups $(Ad_{27n}(C_{27n}(R)), \circ) = (T1_{27n}(C_{27n}(R)), \circ)$, $(V_{27n,3}(C_{27n}(R)), \circ)$ and $(T2_{27n,3}(C_{27n}(R)), \circ)$.

Through-out this paper, for a set $R = \{r_1, r_2, ..., r_k\}$, $C_n(R)$ denotes circulant graph $C_n(r_1, r_2, ..., r_k)$ where $1 \leq r_1 < r_2 < \cdots < r_k \leq [n/2]$. We consider only connected circulant graphs of finite order, $V(C_n(R)) = \{v_0, v_1, v_2, ..., v_{n-1}\}$ with v_i adjacent to v_{i+r} for each $r \in R$, subscript addition taken modulo n and all cycles have length at least 3, unless otherwise specified, $0 \leq i \leq n\text{-}1$. However when $\frac{n}{2} \in R$, edge $v_i v_{i+\frac{n}{2}}$ is taken as a single edge for considering the degree of the vertex v_i or $v_{i+\frac{n}{2}}$ and as a double edge while counting the number of edges or cycles in $C_n(R)$, $0 \leq i \leq n\text{-}1$.

Circulant graph is also defined as a Cayley graph or digraph of a cyclic group. If a graph G is circulant, then its adjacency matrix $A(G)$ is circulant. It follows that if the first row of the adjacency matrix of a circulant graph is $[a_1, a_2, ..., a_n]$, then $a_1 = 0$ and $a_i = a_{n-i+2}$, $2 \leq i \leq n$ [3]. We will often assume, with-out further comment, that the vertices are the corners of a regular *n-gon*, labeled clockwise. Circulant graphs $C_{16}(1,2,7)$ and $C_{16}(2,3,5)$ are shown in Figures 1 and 2, respectively.

Now, we present a few definitions and results that are required in this paper.

Theorem 1.1 [10] *If $C_n(R) \cong C_n(S)$, then there is a bijection f from R to S so that for all $r \in R$, gcd(n, r) = gcd(n, f(r)).* □

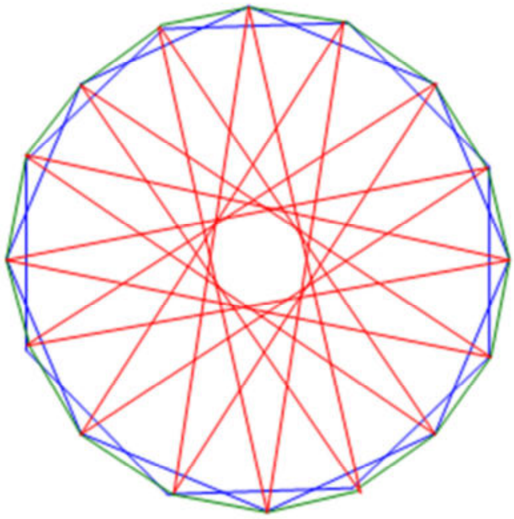

Fig.1. $C_{16}(1,2,7)$

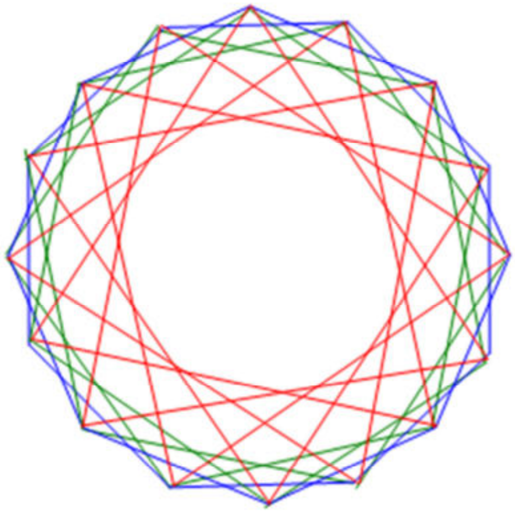

Fig.2. $C_{16}(2,3,5)$

Definition 1.2 [9] A circulant graph $C_n(R)$ is said to have the *CI-property* if whenever $C_n(S)$ is isomorphic to $C_n(R)$, there is some $a \in Z_n^*$ for which $S = aR$.

Lemma 1.3 [13] *Let S be a non-empty subset of* $\mathbb{Z}_n$ *and* $x \in \mathbb{Z}_n$. *Define a mapping* $\Phi_{n,x}$: $S \to \mathbb{Z}_n$ *such that* $\Phi_{n,x}(s) = xs$ *for every* $s \in S$ *under multiplication modulo n. Then* $\Phi_{n,x}$ *is bijective if and only if* $S = \mathbb{Z}_n$ *and* $gcd(n, x) = 1$. □

Definition 1.4 [1] Circulant graphs, $C_n(R)$ and $C_n(S)$ for $R = \{r_1, r_2, \ldots, r_k\}$ and $S = \{s_1, s_2, \ldots, s_k\}$ are *Adam's isomorphic* or *Type*-1 *isomorphic* if there exists a positive integer x relatively prime to n with $S = \{xr_1, xr_2, \ldots, xr_k\}_n^*$ where $< r_i >_n^*$, the *reflexive modular reduction* of a sequence $<r_i>$ is the sequence obtained by reducing each r_i modulo n to yield r_i' and then replacing all resulting terms r_i' which are larger than $\frac{n}{2}$ by n-r_i'.

Lemma 1.5 [13] *Let* $m,r,t \in \mathbb{Z}_n$ *such that* $gcd(n, r) = m > 1$ and $0 \le t \le \frac{n}{m}$-1. *Then the mapping* $\theta_{n,r,t}$: $\mathbb{Z}_n \to \mathbb{Z}_n$ *defined by* $\theta_{n,r,t}(x) = x+jtm$ *for every* $x \in \mathbb{Z}_n$ *under arithmetic modulo n is bijective where* $x = qm+j$, $0 \le j \le m$-1, $0 \le q \le \frac{n}{m}$-1 *and* $j,q \in \mathbb{Z}_n$. □

Theorem 1.6 [13] *Let* $V(C_n(R)) = \{v_0, v_1, v_2, \ldots, v_{n-1}\}$, $V(K_n) = \{u_0, u_1, u_2, \ldots, u_{n-1}\}$, $R = \{r_1, r_2, \ldots, r_k, n - r_k, n - r_{k-1}, \ldots, n - r_1\}$ *and* $r \in R$ *such that* $gcd(n, r) = m > 1$. *Then the mapping* $\theta_{n,r,t}$: $V(C_n(R)) \to V(C_n(1,2,\ldots,n\text{-}1)) = V(K_n)$ *defined by* $\theta_{n,r,t}(v_x) = u_{x+jtm}$ *and* $\theta_{n,r,t}((v_x,v_{x+s})) = (\theta_{n,r,t}(v_x),\theta_{n,r,t}(v_{x+s}))$ *for every* $x \in \mathbb{Z}_n$, $x = qm+j$, $0 \le j \le m$-1, $0 \le q,t \le \frac{n}{m}$-1 *and* $s \in R$, *under subscript arithmetic modulo n, is one-to-one, preserves adjacency and* $\theta_{n,r,t}(C_n(R)) \cong C_n(R)$ *for* $t = 0,1,2,\ldots,\frac{n}{m}$ - 1. □

Definition 1.7 [13] *For a given circulant graph* $C_n(R)$ *and for a particular value of t,* $0 \le t \le \frac{n}{m}$ -1 *if* $\theta_{n,r,t}(C_n(R)) = C_n(S)$ *for some* $S \subseteq [1,\frac{n}{2}]$ *and* $S \ne xR$ *for all* $x \in \phi_n$ *under reflexive modulo n, then* $C_n(R)$ *and* $C_n(S)$ *are called Type-2 isomorphic circulant graphs w.r.t. r,* $r \in R$. *In this case, subsets R and S of* $\mathbb{Z}_n$ *are called Type-2 isomorphic subsets of* $\mathbb{Z}_n$ *w.r.t. r.*

Thus, clearly Type-2 isomorphic circulant graphs are circulant graphs without CI-property.

Theorem 1.8 [13] *For* $n \ge 2$, $k \ge 3$, $1 \le 2s\text{-}1 \le 2n\text{-}1$, $n \ne 2s\text{-}1$, $R = \{2s\text{-}1, 4n\text{-}2s\text{+}1, 2p_1, 2p_2,\ldots,2p_{k-2}\}$ and $S = \{2n\text{-}2s\text{+}1, 2n\text{+}2s\text{-}1, 2p_1,2p_2,\ldots,2p_{k-2}\}$, *circulant graphs* $C_{8n}(R)$ *and* $C_{8n}(S)$ *are Type-2 isomorphic* (*and without CI-property*) *where* $gcd(p_1,p_2,\ldots,p_{k-2}) = 1$ and $n, s, p_1,p_2,\ldots,p_{k-2} \in \mathbb{N}$. □

Theorem 1.9 [13] *For* $R = \{2, 2s\text{-}1, 2s'\text{-}1\}$, $1 \le t \le [\frac{n}{2}]$, $1 \le 2s\text{-}1 < 2s'\text{-}1 \le [\frac{n}{2}]$ *and* $n,s,s',t \in \mathbb{N}$ *if* $C_n(R)$ *and* $\theta_{n,2,t}(C_n(R))$ *are Type-2 isomorphic circulant graphs for some t, then* $n \equiv 0 \ (mod\ 8)$, $2s\text{-}1\text{+}2s'\text{-}1 = \frac{n}{2}$, $t = \frac{n}{8}$ *or* $\frac{3n}{8}$, $2s'\text{-}1 \ne \frac{n}{8}$, $1 \le 2s\text{-}1 \le \frac{n}{4}$ *and* $n \ge 16$. □

Definition 1.10 [13] Let $Ad_n(C_n(R)) = T1_n(C_n(R)) = \{\Phi_{n,x}(C_n(R)): x \in \Phi_n\} = \{C_n(xR): x \in \Phi_n\}$ for a set $R = \{r_1, r_2, \ldots, r_k, n - r_k, n - r_{k-1}, \ldots, n - r_1\}$. Define 'o' in $Ad_n(C_n(R))$ such that $\Phi_{n,x}(C_n(R))$o $\Phi_{n,y}(C_n(R)) = \Phi_{n,xy}(C_n(R))$ and $C_n(xR)$ o $C_n(yR) = C_n((xy)R)$ for every $x,y \in \Phi_n$, under arithmetic modulo n. Clearly, $Ad_n(C_n(R))$ is the set of all circulant graphs which are Adam's isomorphic to $C_n(R)$ and $(Ad_n(C_n(R))$, o) = $(T1_n(C_n(R))$, o) is an abelian group called *the Adam's group* or *the Type*-1 *group on* $C_n(R)$ under 'o'.

Definition 1.11 [13] *Let* $V(C_n(R)) = \{v_0, v_1, v_2, \ldots, v_{n-1}\}$, $V(K_n) = \{u_0, u_1, u_2, \ldots, u_{n-1}\}$, $r \in R$, $m,q,t,t',x \in \mathbb{Z}_n$ *such that* $gcd(n, r) = m > 1$, $x = qm+j$, $0 \le j \le m-1$ and $0 \le q,t,t' \le \frac{n}{m}$ -1. Define $\theta_{n,r,t}$: $\mathbb{Z}_n \to \mathbb{Z}_n$ and $\theta_{n,r,t}$: $V(C_n(R)) \to V(C_n(1,2,\ldots,n\text{-}1)) = V(K_n)$ such that $\theta_{n,r,t}(x) = x+jtm$, $\theta_{n,r,t}(v_x) = u_{x+jtm}$ and $\theta_{n,r,t}((v_x, v_{x+y}))$ = $(\theta_{n,r,t}(v_x)$, $\theta_{n,r,t}(v_{x+y}))$ for every $x \in \mathbb{Z}_n$ and $y \in R$, under subscript arithmetic modulo n. Let $s \in \mathbb{Z}_n$, $V_{n,r}$ = $\{\theta_{n,r,t}$: $t = 0,1,\ldots,\frac{n}{m}$ -1}, $V_{n,r}(s) = \{\theta_{n,r,t}(s)$: $t = 0,1,\ldots,\frac{n}{m}$ -1} and $V_{n,r}(C_n(R)) = \{\theta_{n,r,t}(C_n(R))$: $t = 0,1,\ldots,\frac{n}{m}$ -1}. Define 'o' in $V_{n,r}$ such that $\theta_{n,r,t}\, o\, \theta_{n,r,t'} = \theta_{n,r,t+t'}$, $(\theta_{n,r,t}\, o \theta_{n,r,t'})(x)$ ($= \theta_{n,r,t}(\theta_{n,r,t'}(x)) = \theta_{n,r,t}(x+jt'm) = (x+jt'm)+jtm = x+j(t+t')m$) $= \theta_{n,r,t+t'}(x)$ and $\theta_{n,r,t}(C_n(R))\, o\, \theta_{n,r,t'}(C_n(R)) = \theta_{n,r,t+t'}(C_n(R))$ for every $\theta_{n,r,t}, \theta_{n,r,t'} \in V_{n,r}$ where $t+t'$ is calculated under addition modulo $\frac{n}{m}$. Clearly, ($V_{n,r}(s)$, o) and ($V_{n,r}(C_n(R))$, o) are *abelian groups for all* $s \in \mathbb{Z}_n$.

Properties of $\boldsymbol{\theta_{n,r,t}(C_n(R))}$

1.1 Let $\theta_{n,r,t}(C_n(R)) = C_n(S)$ and $r_i \in \mathbb{Z}_n$ such that $gcd(n,r_i) = gcd(n,r)$. Then, $r_i \in R$ if and only if $r_i \in S$, follows from the definition of $\theta_{n,r,t}$.

1.2 For a given circulant graph $C_n(R)$ and for a particular value of t, if $\theta_{n,r,t}(C_n(R)) = C_n(S)$ for some $S \subseteq [1, [\frac{n}{2}]]$, then $\theta_{n,r,t+t'}(C_n(R)) = \theta_{n,r,t'}(C_n(S))$ for every t', $0 \le t,t' \le \frac{n}{m}$ -1 where $gcd(n, r) = m > 1$. This follows from the fact, $\theta_{n,r,t+t'}(C_n(R)) = \theta_{n,r,t'+t}(C_n(R)) = \theta_{n,r,t'}(\theta_{n,r,t}(C_n(R))) = \theta_{n,r,t'}(C_n(S))$.

1.3 Let $C_n(R)$ and $C_n(S)$ be isomorphic circulant graphs. Then $C_n(S) = \theta_{n,r,t}(C_n(R))$ for some t, $0 \le t \le \frac{n}{m} - 1$ if and only if $C_n(R) = \theta_{n,r,\frac{n}{m}-t}(C_n(S))$. This follows from the fact that $\theta_{n,r,\frac{n}{m}-t}(C_n(S)) = \theta_{n,r,\frac{n}{m}-t}(\theta_{n,r,t}(C_n(R))) = \theta_{n,r,\frac{n}{m}-t+t}(C_n(R)) = \theta_{n,r,0}(C_n(R)) = C_n(R)$ if and only if $C_n(S) = \theta_{n,r,t}(C_n(R))$.

1.4 For isomorphic circulant graphs $C_n(R)$ and $C_n(S)$, $C_n(S) \in T2_{n,r}(C_n(R))$ if and only if $C_n(S) = \theta_{n,r,t}(C_n(R))$ for some t, $0 \le t \le \frac{n}{m} - 1$ and $C_n(R)$ and $C_n(S)$ are Type-2 isomorphic w.r.t. r if and only if $C_n(R) = \theta_{n,r,\frac{n}{m}-1}(C_n(S))$ for some t, $0 \le t \le \frac{n}{m} - 1$ and $C_n(R)$ and $C_n(S)$ are Type-2 isomorphic w.r.t. r if and only if $C_n(R) \in T2_{n,r}(C_n(S))$.

1.5 Let $C_n(R)$, $C_n(S)$ be two isomorphic circulant graphs of Type-2 w.r.t. r, $r \in R,S$ and $R \neq S$. Then, $T2_{n,r}(C_n(R)) = T2_{n,r}(C_n(S))$ follows from Property 1.4.

1.6 Let $C_n(R)$ and $C_n(S)$ be two isomorphic circulant graphs and $R \neq S$. Then, at least one of the following statements is true.

(i) $C_n(S) = C_n(xR)$, $x \in \phi_n$. That is $C_n(R)$ and $C_n(S)$ are Adam's isomorphic.

(ii) $T2_{n,r}(C_n(R)) = T2_{n,r}(C_n(S))$. This implies that $C_n(R)$ and $C_n(S)$ are Type-2 isomorphic circulant graphs w.r.t. r.

(iii) $C_n(S) \neq C_n(xR)$ for all $x \in \phi_n$ and $T2_{n,r}(C_n(R)) \neq T2_{n,r}(C_n(S))$ for any particular $r \in \mathbb{Z}_n$. That is circulant graphs $C_n(R)$ and $C_n(S)$ are neither Adam's isomorphic nor Type-2 isomorphic w.r.t. any particular $r \in \mathbb{Z}_n$. But their isomorphism is connected by a sequence of isomorphic transformations involving Type-2 isomorphisms w.r.t. different r's or Type-2 isomorphisms w.r.t. different r's as well as Adam's isomorphism.

As an example the two circulant graphs $C_{27}(1,3,8,10)$ and $C_{27}(2,7,11,12)$ are isomorphic but they are neither Adam's nor Type-2 isomorphic w.r.t. 3 or 12 (or w.r.t. any particular r whose gcd with 27 is > 1) because of the following.

a) $\phi_{27,x}(C_{27}(1,3,8,10)) \neq C_{27}(2,7,11,12)$ for every $x \in \phi_{27}$ (See Table-1). This implies, $C_{27}(1,3,8,10)$ and $C_{27}(2,7,11,12)$ are not Adam's isomorphic.

b) Even though $gcd(27, 3) = 3 = gcd(27, 12)$, the two circulant graphs $C_{27}(1,3,8,10)$ and $C_{27}(2,7,11,12)$ don't have common jump size, say m, such that $gcd(27, m) = 3$ or $gcd(27, m) = 12$ and so they can't be Type-2 isomorphic w.r.t. any m.

c) $\phi_{27,2}(C_{27}(2,7,11,12)) = \phi_{27,2}(C_{27}(2,7,11,12,15,16,20,25)) = C_{27}(4,14,22,24,30,32,40,50) = C_{27}(4,14,22,24,3,5,13,23) = C_{27}(3,4,5,13)$ which implies that $C_{27}(3,4,5,13)$ and $C_{27}(2,7,11,12)$ are Adam's isomorphic.

d) $\theta_{27,3,1}(C_{27}(1,3,8,10)) = \theta_{27,3,1}(C_{27}(1,3,8,10,17,19,24,26)) = C_{27}(4,3,14,13,23,22,24,32) = C_{27}(4,3,14,13,23,22,24,5) = C_{27}(3,4,5,13)$ which implies, $C_{27}(3,4,5,13) \cong C_{27}(1,3,8,10)$. Also, $\theta_{27,3,2}(C_{27}(1,3,8,10)) = \theta_{27,3,2}(C_{27}(1,3,8,10,17,19,24,26)) = C_{27}(7,3,20,16,2,25,24,11) = C_{27}(2,3,7,11)$. $\theta_{27,3,3}(C_{27}(1,3,8,10)) = \theta_{27,3,3}(C_{27}(1,3,8,10,17,19,24,26)) = C_{27}(10,3,26,19,8,1,24,17) = C_{27}(1,3,8,10)$. Thus, $C_{27}(3,4,5,13) \cong C_{27}(2,7,11,12)$ and $C_{27}(3,4,5,13) \cong C_{27}(1,3,8,10)$ which implies, $C_{27}(1,3,8,10) \cong C_{27}(2,7,11,12)$ but they are not Type-2 isomorphic w.r.t. any particular r.

Thus, we could see that for a given a circulant graph $C_n(R)$ one can make sequence of isomorphic transformations involving Adam's isomorphism as well as Type-2 isomorphisms w.r.t. different r's and obtain an isomorphic circulant graph $C_n(S)$ which may not be Adam's isomorphic or Type-2 isomorphic w.r.t. a particular r to $C_n(R)$. And thus a new study is needed to find the sequence of isomorphisms involved among isomorphic circulant graphs.

Table 1. Calculation of xr under arithmetic modulo 27, $x \in \phi_{27}$ and $r \in R$.

	Jump Size r							
Multiplier x	**1**	**3**	**8**	**10**	**17**	**19**	**24**	**26**
2	2	6	16	20	7	11	21	25
4	4	12	5	13	14	22	15	23
5	5	15	13	23	4	14	12	22
7	7	21	2	16	11	25	6	20
8	8	24	10	26	1	17	3	19
10	10	3	26	19	8	1	24	17
11	11	6	7	2	25	20	21	16
13	13	12	23	22	5	4	15	14

Moreover, $V_{n,r}(C_n(R))$ contains all isomorphic circulant graphs of Type 2 of $C_n(R)$ w.r.t. r, if exist. Let $T2_{n,r}(C_n(R)) = \{C_n(R)\} \cup \{C_n(S)$: $C_n(S)$ is Type-2 isomorphic to $C_n(R)$ w.r.t. $r\}$. Thus, $T2_{n,r}(C_n(R)) = \{C_n(R)\} \cup \{\theta_{n,r,t}(C_n(R))$: $\theta_{n,r,t}(C_n(R)) = C_n(S)$ and $C_n(S)$ is Type-2 isomorphic to $C_n(R)$ w.r.t. r, $0 \le t \le \frac{n}{m}$ - 1$\} \subseteq V_{n,r}(C_n(R))$ and $(T2_{n,r}(C_n(R))$, o) is a subgroup of $(V_{n,r}(C_n(R))$, o) (See Theorem 1.12.). Clearly, $T1_n(C_n(R)) \cap T2_{n,r}(C_n(R)) = \{C_n(R)\}$. $C_n(R)$ has Type-2 isomorphic circulant graph w.r.t. r if and only if $T2_{n,r}(C_n(R)) \neq \{C_n(R)\}$ if and only if $T2_{n,r}(C_n(R)) \cap \{C_n(R)\} \neq \Phi$ if and only if $|T2_{n,r}(C_n(R))| > 1$.

Theorem 1.12 [11] Let $C_n(R)$ be any circulant graph, $r \in R$ and $gcd(n, r) > 1$. Then, $(T2_{n,r}(C_n(R))$, o) is a subgroup of $(V_{n,r}(C_n(R))$, o).

Proof Clearly, $T2_{n,r}(C_n(R)) \subseteq V_{n,r}(C_n(R))$. In $T2_{n,r}(C_n(R))$,$C_n(R) = \theta_{n,r,0}(C_n(R))$. If $T2_{n,r}(C_n(R)) = \{\theta_{n,r,0}(C_n(R)) = C_n(R)\}$, then $(T2(\theta_{n,r,0}(C_n(R)))$, o) is a group that contains identity element only.

If $T2_{n,r}(C_n(R)) \neq \{\theta_{n,r,0}(C_n(R)) = C_n(R)\}$, then let $C_n(S) \in T2_{n,r}(C_n(R))$ with $R \neq S$. This implies, $C_n(S) = \theta_{n,r,t}(C_n(R))$ for some t and $C_n(R)$ and $C_n(S)$ are Type-2 isomorphic w.r.t. r, $1 \le t \le \frac{n}{m}$ -1. And $T2_{n,r}(C_n(R)) = T2_{n,r}(C_n(S))$, $R \neq S$ using the Property 1.5.

This implies, for $1 \le t,t' \le \frac{n}{m}$ -1 and $R \neq S$, $\theta_{n,r,t}(C_n(R)) = C_n(S)$ and $C_n(R) = \theta_{n,r,t'}(C_n(S)) = \theta_{n,r,t'}(\theta_{n,r,t}(C_n(R))) = \theta_{n,r,t'+t}(C_n(R)) = \theta_{n,r,t'}(C_n(R))$ o $\theta_{n,r,t}(C_n(R))$, using the definition of $\theta_{n,r,t}$. This implies, $\theta_{n,r,t'}(C_n(R))$o$\theta_{n,r,t}(C_n(R)) = C_n(R) = \theta_{n,r,o}(C_n(R))$, using the definition of $\theta_{n,r,t'}$,$\theta_{n,r,t}(C_n(R)) = C_n(S)$,$\theta_{n,r,t'}(C_n(S)) = C_n(R) \in T2_{n,r}(C_n(R))$, $0 \le t,t' \le \frac{n}{m}$ -1. This implies that $t+t' \equiv 0$ ($mod \frac{n}{m}$) and also $\theta_{n,r,t'}(C_n(R))$ and $\theta_{n,r,t}(C_n(R))$ are inverse elements in $(T2_{n,r}(C_n(R))$, o) which implies that $C_n(S)$ and $\theta_{n,r,t'}(C_n(R))$ are inverse elements in $(T2_{n,r}(C_n(R))$, o) for some t', $1 \le t,t' \le \frac{n}{m}$ -1 and $t+t' \equiv 0$ (mod $\frac{n}{m}$). This implies, $t' = \frac{n}{m} - t$ and $\theta_{n,r,t'}(C_n(R)) \in T2_{n,r}(C_n(R))$, $1 \le t,t' \le \frac{n}{m}$ -1.

Also, we have if $C_n(R)$ and $\theta_{n,r,t}(C_n(R))$ are Type-2 isomorphic for a particular t, then $C_n(R)$ and $\theta_{n,r,\frac{n}{m}-t}(C_n(R))$ are also Type-2 isomorphic circulant graphs. This implies, $\theta_{n,r,t'}(C_n(R)) \in T2_{n,r}(C_n(R))$ and hence $C_n(S)$ and $\theta_{n,r,t'}(C_n(R))$ are inverse elements in $(T2_{n,r}(C_n(R))$, o) for some t' where $1 \le t,t' \le \frac{n}{m}$ - 1 and $t+t' \equiv 0$ (mod $\frac{n}{m}$).

Other laws of Abelian group are easy to prove. Hence the result follows. □

Definition 1.13 [15] For any circulant graph $C_n(R)$, if group $(T2_{n,r}(C_n(R))$, o) exists, then it is called *the Type-2 group of* $C_n(R)$ *w.r.t. r* under 'o'.

Theorem 1.14 [14] For $n \ge 2$, $k \ge 3$, $1 \le 2s\text{-}1 \le 2n\text{-}1$, $n \neq 2s\text{-}1$, $R = \{2s\text{-}1, 4n\text{-}2s\text{+}1, 2p_1, 2p_2,...,2p_{k-2}\}$ and $S = \{2n\text{-}(2s\text{-}1), 2n\text{+}2s\text{-}1, 2p_1, 2p_2,...,2p_{k-2}\}$, $T2_{8n,2}(C_{8n}(R)) = T2_{8n,2}(C_{8n}(S))$, $(T2_{8n,2}(C_{8n}(R))$, o) = $(T2_{8n,2}(C_{8n}(S))$, o) is a Type-2 group of order 2 and $(T2_{8n,2}(C_{8n}(R \cup 8n - R))$, o) = $(T2_{8n,2}(C_{8n}(S \cup 8n - S))$, o) where $gcd(p_1,p_2,...,p_{k-2}) = 1$ and $n, s, p_1,p_2,...,p_{k-2} \in \mathbb{N}$. □

Obtaining new families of circulant graphs without CI-property is the motivation for this work. For all basic ideas in graph theory, we follow **[5]**.

2 Family of Type-2 Isomorphic Circulant Graphs and Abelian Groups

Theorem 2.1 *For* $n \in \mathbb{N}$, $R = \{1, 3, 9n\text{-}1, 9n\text{+}1\}$, $S = \{3, 3n\text{+}1, 6n\text{-}1, 12n\text{+}1\}$ *and* $T = \{3, 3n\text{-}1, 6n\text{+}1, 12n\text{-}1\}$, $C_{27n}(R)$, $C_{27n}(S)$ *and* $C_{27n}(T)$ *are isomorphic circulant graphs.*

Proof: Here, we prove, $\theta_{27n,3,n}(C_{27n}(R)) = C_{27n}(S)$ and $\theta_{27n,3,2n}(C_{27n}(R)) = C_{27n}(T)$ when $R = \{1, 3, 9n\text{-}1, 9n\text{+}1\}$, $S = \{3, 3n\text{+}1, 6n\text{-}1, 12n\text{+}1\}$ and $T = \{3, 3n\text{-}1, 6n\text{+}1, 12n\text{-}1\}$. To simplify our calculation let us consider $R = \{1, 3, 9n\text{-}1, 9n\text{+}1, 18n\text{-}1, 18n\text{+}1, 27n\text{-}3, 27n\text{-}1\}$, $S = \{3, 3n\text{+}1, 6n\text{-}1, 12n\text{+}1, 15n\text{-}1, 21n\text{+}1, 24n\text{-}1, 27n\text{-}3\}$ and $T = \{3, 3n\text{-}1, 6n\text{+}1, 12n\text{-}1, 15n\text{+}1, 21n\text{-}1, 24n\text{+}1, 27n\text{-}3\}$.

Clearly, $\theta_{n,r,t}: V(C_n(R)) \to V(K_n)$ is a bijective function and by the definition of $\theta_{n,r,t}$, we get $\theta_{27n,3,n}(3) = 3$, $\theta_{27n,3,n}(27n-3) = 27n\text{-}3$, $\theta_{27n,3,n}(1) = 3n\text{+}1$, $\theta_{27n,3,n}(9n\text{+}1) = 12n\text{+}1$, $\theta_{27n,3,n}(18n\text{+}1) = 21n\text{+}1$, $\theta_{27n,3,n}(9n\text{-}1) = 15n\text{-}1$, $\theta_{27n,3,n}(18n\text{-}1) = 24n\text{-}1$ and $\theta_{27n,3,n}(27n-1) = 6n\text{-}1$. This implies, $\theta_{27n,3,2n}(C_{27n}(R)) = C_{27n}(S)$ and $C_{27n}(R) \cong C_{27n}(S)$.

Similarly, $\theta_{27n,3,2n}(3) = 3$, $\theta_{27n,3,2n}(27n-3) = 27n\text{-}3$, $\theta_{27n,3,2n}(1) = 6n\text{+}1$, $\theta_{27n,3,2n}(9n\text{+}1) = 15n\text{+}1$, $\theta_{27n,3,2n}(18n\text{+}1) = 24n\text{+}1$, $\theta_{27n,3,2n}(9n\text{-}1) = 21n\text{-}1$, $\theta_{27n,3,2n}(18n\text{-}1) = 3n\text{-}1$ and $\theta_{27n,3,2n}(27n-1) = 12n\text{-}1$. This implies, $\theta_{27n,3,2n}(C_{27n}(R)) = C_{27n}(T)$ and $C_{27n}(R) \cong C_{27n}(T)$. This implies that $C_{27n}(R) \cong C_{27n}(S) \cong C_{27n}(T)$. Hence the result. □

Theorem 2.2 *For* $n \in \mathbb{N}$, $R = \{1, 3, 9n\text{-}1, 9n\text{+}1\}$, $S = \{3, 3n\text{+}1, 6n\text{-}1, 12n\text{+}1\}$ *and* $T = \{3, 3n\text{-}1, 6n\text{+}1, 12n\text{-}1\}$, $\theta_{27n,3,n}(C_{27n}(R)) = C_{27n}(S)$, $\theta_{27n,3,n}(C_{27n}(S)) = C_{27n}(T)$ *and* $\theta_{27n,3,n}(C_{27n}(T)) = C_{27n}(R)$ *and* $C_{27n}(R)$, $C_{27n}(S)$ *and* $C_{27n}(T)$ *are Type-2 isomorphic circulant graphs.*

Proof: *For* $n \in \mathbb{N}$, $R = \{1, 3, 9n\text{-}1, 9n\text{+}1\}$, $S = \{3, 3n\text{+}1, 6n\text{-}1, 12n\text{+}1\}$ *and* $T = \{3, 3n\text{-}1, 6n\text{+}1, 12n\text{-}1\}$, $\theta_{27n,3,n}(C_{27n}(R)) = C_{27n}(S)$, $\theta_{27n,3,n}(C_{27n}(S)) = C_{27n}(T)$, $\theta_{27n,3,n}(C_{27n}(T)) = C_{27n}(R)$ and $C_{27n}(R) \cong C_{27n}(S) \cong C_{27n}(T)$ using Theorem 2.1. Also, for a given $n \in \mathbb{N}$, the set of jump sizes of the three circulant graphs are different. Here, $R \cap S = \{3\}$ and so if $C_{27n}(R)$ and $C_{27n}(S)$ are Type-2 isomorphic, then they are Type-2 isomorphic w.r.t. $m = 3$ only.

Claim: For $R = \{1, 3, 9n\text{-}1, 9n\text{+}1\}$, $S = \{3, 3n\text{+}1, 6n\text{-}1, 12n\text{+}1\}$ and $n \in \mathbb{N}$, $C_{27n}(R)$ *and* $C_{27n}(S)$ *are Type-2 isomorphic w.r.t.* $m = 3$.

If not, they are of Adam's isomorphic. This implies, there exists $s \in \mathbb{N}$ such that $gcd(27n, s) = 1$ and $C_{27n}(sR) = C_{27n}(S)$ where $s = 3x\text{-}2$ or $s = 3x\text{-}1$, $x \in \mathbb{N}$. Now, let $s = 3x\text{-}2$ such that $gcd(27n, 3x\text{-}2) = 1$, $C_{27n}((3x-2)R) = C_{27n}(S)$ and $s \in \mathbb{N}$. This implies, $(3x\text{-}2)\{1, 3, 9n\text{-}1, 9n\text{+}1, 18n\text{-}1, 18n\text{+}1, 27n\text{-}3, 27n\text{-}1\} = \{3, 3n\text{+}1, 6n\text{-}1, 12n\text{+}1, 15n\text{-}1, 21n\text{+}1, 24n\text{-}1, 27n\text{-}3\}$, under arithmetic modulo $27n$. This implies, $3(3x\text{-}2)$, $(3x\text{-}2)(27n\text{-}3)$, $3\text{+}27np_1$ and $27n\text{-}3\text{+}27np_2$ are the only numbers, each is a multiple of 3, in the two sets for some $p_1, p_2 \in \mathbb{N}_0$. Thus the following two cases arise.

Case i. $3(3x\text{-}2) = 3\text{+}27np_1$, $p_1 \in \mathbb{N}_0$, $1 \le 3x\text{-}2 \le 27n\text{-}1$.

In this case, $p_1 = 0$ or 1 or 2 since $1 \le 3x\text{-}2 \le 27n\text{-}1$ and $n, x \in \mathbb{N}$. When $p_1 = 0$, $3x\text{-}2 = 1$; $p_1 = 1$, $3x\text{-}2 = 9n\text{+}1$; $p_1 = 2$, $3x\text{-}2 = 18n\text{+}1$ and in each case, the two graphs are the same. The jump sizes of the circulant graph corresponding to Adam's isomorphism when $s = 3x\text{-}2 = 9n\text{+}1$ and $s = 3x\text{-}2 = 18n\text{+}1$ are given in Table 2.

Case ii. $3(3x\text{-}2) = 27n\text{-}3\text{+}27np_2$, $p_2 \in \mathbb{N}_0$, $x \in \mathbb{N}$, $1 \le 3x\text{-}2 \le 27n\text{-}1$.

In this case, $p_2 = 0$ or 1 or 2 since $1 \le 3x\text{-}2 \le 27n\text{-}1$ and $n, x \in \mathbb{N}$. When $p_2 = 0$, $3x\text{-}2 = 9n\text{-}1$; $p_2 = 1$, $3x\text{-}2 = 18n\text{-}1$; $p_2 = 2$, $3x\text{-}2 = 27n\text{-}1$ and in each case, the two graphs are the same. The jump sizes of the circulant graph corresponding to Adam's isomorphism when $s = 3x\text{-}2 = 9n\text{-}1$, $s = 3x\text{-}2 = 18n\text{-}1$ and $s = 3x\text{-}2 = 27n\text{-}1$ are given in Table 2.

Table 2. Calculation of rs under arithmetic modulo $27n$ where s = $3x$-2 or $3x$-1

	Jump Size *r*					
Multiplier *s*	**1**	**9*n*-1**	**9*n*+1**	**18*n*-1**	**18*n*+1**	**27*n*-1**
9*n*-1	9*n*-1	9*n*+1	27*n*-1	1	18*n*-1	18*n*+1
9*n*+1	9*n*+1	27*n*-1	18*n*+1	9*n*-1	1	18*n*-1
18*n*-1	18*n*-1	1	9*n*-1	18*n*+1	27*n*-1	9*n*+1
18*n*+1	18*n*+1	18*n*-1	1	27*n*-1	9*n*+1	9*n*-1
27*n*-1	27*n*-1	18*n*+1	18*n*-1	9*n*+1	9*n*-1	1

Now, consider the case when $s = 3x\text{-}1$ with $gcd(27n, 3x\text{-}1) = 1$, $C_{27n}(sR) = C_{27n}(S)$ and $x \in \mathbb{N}$. This implies, $(3x\text{-}1)\{1, 3, 9n\text{-}1, 9n\text{+}1, 18n\text{-}1, 18n\text{+}1, 27n\text{-}3, 27n\text{-}1\} = \{3, 3n\text{+}1, 6n\text{-}1, 12n\text{+}1, 15n\text{-}1, 21n\text{+}1, 24n\text{-}1, 27n\text{-}3\}$, under arithmetic modulo $27n$. This implies, $3(3x\text{-}1)$, $(3x\text{-}1)(27n\text{-}3)$, $3\text{+}27np_1$ and $27n\text{-}3\text{+}27np_2$ are the only numbers, each multiple of 3, in the two sets for some $p_1, p_2 \in \mathbb{N}_0$. The following two cases arise.

Case i. $3(3x\text{-}1) = 3+27np_1$, $p_1 \in \mathbb{N}_0$, $x \in \mathbb{N}$, $1 \le 3x\text{-}1 \le 27n\text{-}1$.
In this case, $p_1 = 0$ or 1 or 2 since $1 \le 3x\text{-}1 \le 27n\text{-}1$ and $n,x \in \mathbb{N}$. When $p_1 = 0$, $3x\text{-}1 = 1$; $p_1 = 1$, $3x\text{-}1 = 9n+1$; $p_1 = 2$, $3x\text{-}1 = 18n+1$ and in each case, $C_{27n}(sR) = C_{27n}((3x-1)R) = C_{27n}(S)$. The jump sizes of the circulant graph corresponding to Adam's isomorphism when $s = 3x\text{-}1 = 9n+1$ and $s = 3x\text{-}1 = 18n+1$ are given in Table 2.
In this case, $p_2 = 0$ or 1 or 2 since $1 \le 3x\text{-}1 \le 27n\text{-}1$ and $n,x \in \mathbb{N}$. When $p_2 = 0$, $3x\text{-}1 = 9n\text{-}1$; $p_2 = 1$, $3x\text{-}1 = 18n\text{-}1$; $p_2 = 2$, $3x\text{-}1 = 27n\text{-}1$ and in each case, $C_{27n}(sR) = C_{27n}((3x-1)R) = C_{27n}(S)$. The jump sizes of the circulant graph corresponding to Adam's isomorphism when $s = 3x\text{-}1 = 9n\text{-}1$, $s = 3x\text{-}1 = 18n\text{-}1$ and $s = 3x\text{-}1 = 27n\text{-}1$ are given in Table 2.
Case ii. $3(3x\text{-}1) = 27n\text{-}3+27np_2$, $p_2 \in \mathbb{N}_0$, $x \in \mathbb{N}$, $1 \le 3x\text{-}1 \le 27n\text{-}1$.
This shows that the isomorphic circulant graphs $C_{27n}(R)$ and $C_{27n}(S)$ for $R = \{1, 3, 9n\text{-}1, 9n+1\}$ and $S = \{3, 3n+1, 6n\text{-}1, 12n+1\}$ are not of Type-1, $n \in \mathbb{N}$.
Now consider isomorphic circulant graphs $C_{27n}(S)$ and $C_{27n}(T)$ for $S = \{3, 3n+1, 6n\text{-}1, 12n+1\}$ and $T = \{3, 3n\text{-}1, 6n+1, 12n\text{-}1\}$, $n \in \mathbb{N}$. Here, $S \cap T = \{3\}$ and so if $C_{27n}(S)$ and $C_{27n}(T)$ are Type-2 isomorphic, then they are Type-2 isomorphic circulant graphs w.r.t. $m = 3$ only.
Claim: For $n \in \mathbb{N}$, $S = \{3, 3n+1, 6n\text{-}1, 12n+1\}$ and $T = \{3, 3n\text{-}1, 6n+1, 12n\text{-}1\}$, *$C_{27n}(S)$ and $C_{27n}(T)$ are Type-2 isomorphic.*
If not, they are of Adam's isomorphic. This implies, there exists $s \in \mathbb{N}$ such that $gcd(27n, s) = 1$ and $C_{27n}(sS) = C_{27n}(T)$ where $s = 3x\text{-}2$ or $s = 3x\text{-}1$, $x \in \mathbb{N}$. Now, let $s = 3x\text{-}2$ such that $gcd(27n, 3x\text{-}2) = 1$, $C_{27n}(sS) = C_{27n}((3x-2)S) = C_{27n}(T)$, $x \in \mathbb{N}$. This implies, $(3x\text{-}2)\{3, 3n+1, 6n\text{-}1, 12n+1, 15n\text{-}1, 21n+1, 24n\text{-}1, 27n\text{-}3\} = \{3, 3n\text{-}1, 6n+1, 12n\text{-}1, 15n+1, 21n\text{-}1, 24n+1, 27n\text{-}3\}$, under arithmetic modulo $27n$. Now, $3(3x\text{-}2)$, $(3x\text{-}2)(27n\text{-}3)$, $3+27np_1$ and $27n\text{-}3+27np_2$ are the only numbers, each is a multiple of 3, in the two sets for some $p_1, p_2 \in \mathbb{N}_0$. Thus the following two cases arise.
Case i. $3(3x\text{-}2) = 3+27np_1$, $p_1 \in \mathbb{N}_0$, $x \in \mathbb{N}$, $1 \le 3x\text{-}2 \le 27n\text{-}1$.
In this case, $p_1 = 0$ or 1 or 2 since $1 \le 3x\text{-}2 \le 27n\text{-}1$ and $n,x \in \mathbb{N}$. This implies, when $p_1 = 0$, $3x\text{-}2 = 1$; $p_1 = 1$, $3x\text{-}2 = 9n+1$; $p_1 = 2$, $3x\text{-}2 = 18n+1$ and in each case, $C_{27n}(sS) = C_{27n}((3x-2)S) = C_{27n}(T)$. The jump sizes of the circulant graph corresponding to Adam's isomorphism when $s = 3x\text{-}2 = 9n+1$ and $s = 3x\text{-}2 = 18n+1$ are given in Table 3.
Case ii. $3(3x\text{-}2) = 27n\text{-}3+27np_2$, $p_2 \in \mathbb{N}_0$, $x \in \mathbb{N}$, $1 \le 3x\text{-}2 \le 27n\text{-}1$.
In this case, $p_2 = 0$ or 1 or 2 since $1 \le 3x\text{-}2 \le 27n\text{-}1$ and $n,x \in \mathbb{N}$. When $p_2 = 0$, $3x\text{-}2 = 9n\text{-}1$; $p_2 = 1$, $3x\text{-}2 = 18n\text{-}1$; $p_2 = 2$, $3x\text{-}2 = 27n\text{-}1$ and in each case, $C_{27n}(sS) = C_{27n}((3x-2)S) = C_{27n}(T)$. The jump sizes of the circulant graph corresponding to Adam's isomorphism when $s = 3x\text{-}2 = 9n\text{-}1$, $s = 3x\text{-}2 = 18n\text{-}1$ and $s = 3x\text{-}2 = 27n\text{-}1$ are given in Table 3.

Table 3. Calculation of rs under arithmetic modulo $27n$ where $s = 3x - 2$ or $3x - 1$.

	Jump Size r					
Multiplier s	$3n+1$	$6n$-1	$12n+1$	$15n$-1	$21n+1$	$24n$-1
$9n$-1	$6n$-1	$12n+1$	$24n$-1	$3n+1$	$15n$-1	$21n+1$
$9n+1$	$12n+1$	$24n$-1	$21n+1$	$6n$-1	$3n+1$	$15n$-1
$18n$-1	$15n$-1	$3n+1$	$6n$-1	$21n+1$	$24n$-1	$12n+1$
$18n+1$	$21n+1$	$15n$-1	$3n+1$	$24n$-1	$12n+1$	$6n$-1
$27n$-1	$24n$-1	$21n+1$	$15n$-1	$12n+1$	$6n$-1	$3n+1$

This shows that the isomorphic circulant graphs $C_{27n}(R)$and $C_{27n}(S)$ for $R = \{1, 3, 9n\text{-}1, 9n+1\}$ and $S = \{3, 3n+1, 6n\text{-}1, 12n+1\}$ are not of Type-1, $n \in \mathbb{N}$.
Now consider the case when $s = 3x\text{-}1$ with $gcd(27n, 3x\text{-}1) = 1$, $C_{27n}((3x-1)S) = C_{27n}(T)$ and $x \in \mathbb{N}$. This implies, $(3x\text{-}1)\{3, 3n+1, 6n\text{-}1, 12n+1, 15n\text{-}1, 21n+1, 24n\text{-}1, 27n\text{-}3\} = \{3, 3n\text{-}1, 6n+1, 12n\text{-}1, 15n+1, 21n\text{-}1, 24n+1, 27n\text{-}3\}$, under arithmetic modulo $27n$. This implies, $3(3x\text{-}1)$, $(3x\text{-}1)(27n\text{-}3)$, $3+27np_1$ and $27n\text{-}3+27np_2$ are the only numbers, each is a multiple of 3, in the two sets for some $p_1, p_2 \in \mathbb{N}_0$. The following two cases arise.
Case i. $3(3x\text{-}1) = 3+27np_1$, $p_1 \in \mathbb{N}_0$, $x \in \mathbb{N}$, $1 \le 3x\text{-}1 \le 27n\text{-}1$.
In this case, $p_1 = 0$ or 1 or 2 since $1 \le 3x\text{-}1 \le 27n\text{-}1$ and $n,x \in \mathbb{N}$. When $p_1 = 0$, $3x\text{-}1 = 1$; $p_1 = 1$, $3x\text{-}1 = 9n+1$; $p_1 = 2$, $3x\text{-}1 = 18n+1$ and in each case, $C_{27n}(sS) = C_{27n}((3x-1)S) = C_{27n}(T)$. The jump sizes of the circulant graph corresponding to Adam's isomorphism when $s = 3x\text{-}1 = 9n+1$ and $s = 3x\text{-}1 = 18n+1$ are given in Table 3.
Case ii. $3(3x\text{-}1) = 27n\text{-}3+27np_2$, $p_2 \in \mathbb{N}_0$, $x \in \mathbb{N}$, $1 \le 3x\text{-}1 \le 27n\text{-}1$.

In this case, $p_2 = 0$ or 1 or 2 since $1 \leq 3x\text{-}1 \leq 27n\text{-}1$ and $n,x \in \mathbb{N}$. When $p_2 = 0$, $3x\text{-}1 = 9n\text{-}1$; $p_2 = 1$, $3x\text{-}1 = 18n\text{-}1$; $p_2 = 2$, $3x\text{-}1 = 27n\text{-}1$ and in each case, $C_{27n}(sS) = C_{27n}((3x-1)S) = C_{27n}(T)$. The jump sizes of the circulant graph corresponding to Adam's isomorphism when $s = 3x\text{-}1 = 9n\text{-}1$, $s = 3x\text{-}1 = 18n\text{-}1$ and $s = 3x\text{-}1 = 27n\text{-}1$ are given in Table 3.

This shows that the isomorphic circulant graphs $C_{27n}(S)$ and $C_{27n}(T)$ for $S = \{3, 3n+1, 6n\text{-}1, 12n+1\}$ and $T = \{3, 3n\text{-}1, 6n+1, 12n\text{-}1\}$ are not of Type-1, $n \in \mathbb{N}$.

Similarly, we can prove that isomorphic circulant graphs $C_{27n}(R)$ and $C_{27n}(T)$ for $R = \{1, 3, 9n\text{-}1, 9n+1\}$ and $T = \{3, 3n\text{-}1, 6n+1, 12n\text{-}1\}$ are not of Type-1, $n \in \mathbb{N}$.

Thus, all the 3 different isomorphic circulant graphs $C_{27n}(R)$, $C_{27n}(S)$ and $C_{27n}(T)$ for $R = \{1, 3, 9n\text{-}1, 9n+1\}$, $S = \{3, 3n+1, 6n\text{-}1, 12n+1\}$ and $T = \{3, 3n\text{-}1, 6n+1, 12n\text{-}1\}$ are not of Type-1. Moreover, $\theta_{27n,3,n}(C_{27n}(R)) = C_{27n}(S)$, $\theta_{27n,3,n}(C_{27n}(S)) = C_{27n}(T)$ *and* $\theta_{27n,3,n}(C_{27n}(T)) = C_{27n}(R)$, $n \in \mathbb{N}$. Hence the result follows. □

Theorem 2.3 For $k \geq 3$, $R = \{1, 9n\text{-}1, 9n+1, 3p_1, 3p_2, \ldots, 3p_{k-2}\}$, $S = \{3n+1, 6n\text{-}1, 12n+1, 3p_1, 3p_2, \ldots, 3p_{k-2}\}$ and $T = \{3n\text{-}1, 6n+1, 12n\text{-}1, 3p_1, 3p_2, \ldots, 3p_{k-2}\}$, circulant graphs $C_{27n}(R)$, $C_{27n}(S)$ and $C_{27n}(T)$ are Type-2 isomorphicwith $m_i = 3$ and without CI-property where $gcd(p_1, p_2, \ldots, p_{k-2}) = 1$ and $n, p_1, p_2, \ldots, p_{k-2} \in \mathbb{N}$.

Proof: When $R = \{1, 3, 9n\text{-}1, 9n+1\}$, $S = \{3, 3n+1, 6n\text{-}1, 12n+1\}$ and $T = \{3, 3n\text{-}1, 6n+1, 12n\text{-}1\}$, $C_{27n}(R)$, $C_{27n}(S)$ and $C_{27n}(T)$ are Type-2 isomorphic circulant graphs, using Theorem 2.2, $n \in \mathbb{N}$. Lemma 1.5 helps us while searching for possible value(s) of t such that the transformed graph $\theta_{n,r,t}(C_n(R))$ is circulant of the form $C_{27n}(S)$ for some $S \subseteq [1, \frac{n}{2}]$, the calculation on r_js which are integer multiples of $m = gcd(n, r)$ need not be done as there is no change in these r_js under the transformation $\theta_{n,r,t}$. This implies when $R = \{1, 9n\text{-}1, 9n+1, 3p_1, 3p_2, \ldots, 3p_{k-2}\}$, $S = \{3n+1, 6n\text{-}1, 12n+1, 3p_1, 3p_2, \ldots, 3p_{k-2}\}$ and $T = \{3n\text{-}1, 6n+1, 12n\text{-}1, 3p_1, 3p_2, \ldots, 3p_{k-2}\}$, circulant graphs $C_{27n}(R)$, $C_{27n}(S)$ and $C_{27n}(T)$ are Type-2 isomorphic where $k \geq 3$, $gcd(p_1, p_2, \ldots, p_{k-2}) = 1$ and $n, p_1, p_2, \ldots, p_{k-2} \in \mathbb{N}$. Type-2 isomorphic circulant graphs are graphs without CI-property. Hence the result follows. □

Type 2 isomorphic circulant graphs $C_{27}(1,3,8,10)$, $C_{27}(3,4,5,13)$ and $C_{27}(2,3,7,11)$ are given in Figures 3,4,5, respectively.

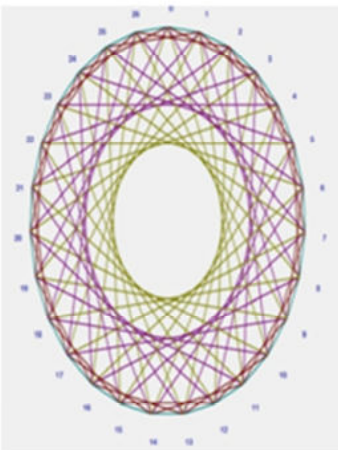

Fig.3. $C_{27}(1,3,8,10)$.

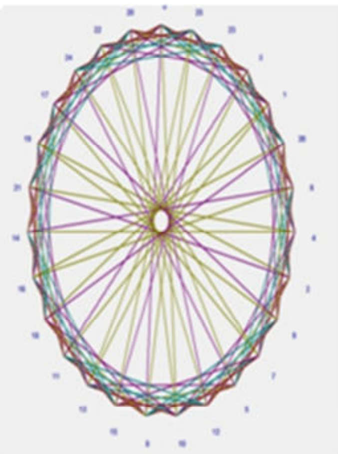

Fig.4. $C_{27}(3,4,5,13)$.

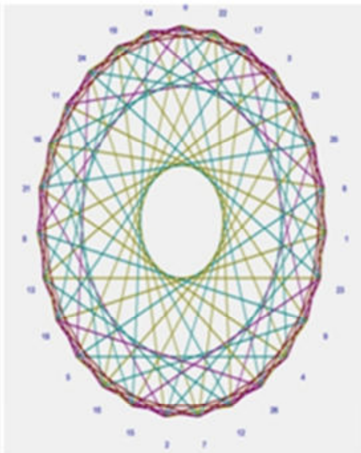

Fig.5. $C_{27}(2,3,7,11)$

II. Conclusion

The results derived in this paper and in [13] on circulant graphs of Type-2 isomorphism and without CI-property are based on circulant graphs with three and two copies of isomorphic circulant subgraphs, respectively. One can try similar results on circulant graphs with $m = gcd(n, r)$ is odd and > 3.

Acknowledgement

We express our sincere thanks to Prof. M. I. Jinnah, University of Kerala, Trivandrum, India and Prof. V. Mohan, Thiyagarayar College of Engineering, Madurai, Tamil Nadu, India for their valuable suggestions and guidance. We also express our gratitude to Central University of Kerala, Kasaragod, Kerala, India and St. Jude's College, Thoothoor, Kanyakumari District, Tamil Nadu, India for providing facilities to do this research work.

References

[1] A. Adam, Research problem 2–10,*J. Combinatorial Theory,* **3** (1967), 393.
[2] B. Alspach, J. Morris and V. Vilfred, Self-complementary circulant graphs, *Ars Com.,* **53** (1999), 187-191.
[3] P.J. Davis, Circulant Matrices, *Wiley, New York,* 1979.
[4] B. Elspas and J. Turner, Graphs with circulant adjacency matrices, *J. Combinatorial Theory,* **9** (1970), 297-307.
[5] F. Harary, Graph Theory, *Addison-Wesley, Reading, MA,* 1969.
[6] I. Kra and S. R. Simanca, On Circulant Matrices, AMS Notices, **59** (2012), 368-377.
[7] C. H. Li, On isomorphisms of finite Cayley graphs - a survey. Discrete Math. **256** (2002), 301–334.
[8] J. Morris, Automorphism groups of circulant graphs – a survey, arXiv: math/ 0411302v1 [math.CO], 13 Nov. 2004.
[9] M. Muzychuk, On Adam's Conjecture for circulant graphs, *Discrete Math*., **167/168** (1997), 497-510.
[10] V. Vilfred, ∑-labelled Graphs and Circulant Graphs*, Ph.D. Thesis,* University of Kerala, Thiruvananthapuram, India, March 1994.
[11] V. Vilfred, *A study on isomorphic properties of circulant graphs*, Advances in Science, Technology and Engineering Systems *Journal* (ASTES) Journal **Vol. 2** (**6**) (2017), 236-241. DOI: 10.25046/aj020628.
[12] V. Vilfred, A Theory of Cartesians Product and Factorization of Circulant Graphs, *Hindawi Pub. Corp. – J. Discrete Math.*, **Vol. 2013**, Article ID 163740, 10 pages.
[13] V. Vilfred, New Abelian Groups from Isomorphism of Circulant Graphs, *Proce. of Inter. Conf. on Applied Math. and Theoretical Computer Sci.,* St. Xavier's Catholic Engineering College, Nagercoil, Tamil Nadu, India (2013), xiii-xvi. ISBN 978-93-82338-30-7.
[14] V. Vilfred, New Family of Circulant Graphs without Cayley Isomorphism Property with m_i = 2 (Communicated for publication).
[15] V. Vilfred and P. Wilson, New Family of Circulant Graphs without Cayley Isomorphism Property with m_i = 5, *Int. Journal of Scientific and Innovative Mathematical Research*, **3** (**6**) (2015), 39 - 47.
[16] V. Vilfred and P. Wilson, New Family of Circulant Graphs without Cayley Isomorphism Property with m_i = 7,*IOSR Journal of Mathematics*, **12** (2016), 32 - 37.

V. Vilfred, " Circulant Graphs Without Cayley Isomorphism Property With m = 3." IOSR Journal of Mathematics (IOSR-JM) 15.2 (2019): 24-31.