Honam Mathematical J. **38** (2016), No. 1, pp. 25–37
http://dx.doi.org/10.5831/HMJ.2016.38.1.25

ON BASIC ANALOGUE OF CLASSICAL SUMMATION THEOREMS DUE TO ANDREWS

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Abstract. In 1972, Andrews derived the basic analogue of Gauss's second summation theorem and Bailey's theorem by implementing basic analogue of Kummer's theorem into identity due to Jackson. Recently Lavoie *et.al.* derived many results closely related to Kummer's theorem, Gauss's second summation theorem and Bailey's theorem and also Rakha *et. al.* derive the basic analogues of results closely related Kummer's theorem. The aim of this paper is to derive basic analogues of results closely related Gauss's second summation theorem and Bailey's theorem. Some applications and limiting cases are also considered.

1. Introduction

The q-analog of Gauss's hypergeometric function is given by ([6]):

$$(1) \qquad {}_2\phi_1\left[\begin{array}{c} a,b \\ c \end{array};q,z\right]=\sum_{0}^{\infty}\frac{(a;q)_n(b;q)_n}{(q;q)_n(c;q)_n}\frac{z^n}{n!},$$

where

$$(2) \qquad (a;q)_n=\begin{cases} 1;\ n=0 \\ (1-a)(1-aq)\cdots(1-aq^{n-1});\ n\neq 0. \end{cases}$$

The q-analog of generalized hypergeometric function defined as follows (see [5]):

Received May 14, 2015. Accepted December 23, 2015.
2010 Mathematics Subject Classification. 33C05, 33D15.
Key words and phrases: Basic hypergeometric function, generalized Gauss's second summation theorem, generalized Bailey's theorem, Kummer's theorem, Jackson's identity.
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$$
\begin{aligned}
&{}_r\phi_s\left[\begin{array}{c} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{array};q,z\right] = \\
&\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r}(a_i;q)_n}{(q;q)_n\prod_{j=1}^{s}(b_j;q)_n}\left\{(-1)^n q^{\binom{n}{2}}\right\}^{1+s-r}z^n.
\end{aligned}
\tag{3}
$$

Bailey [2] and Daum [3] independently discovered the following summation formula:

$$
{}_2\phi_1\left[\begin{array}{c} a,\ b, \\ \frac{aq}{b}, \end{array};q,-\frac{q}{b}\right] = \frac{(aq;q^2)_\infty(-q;q)_\infty\left(\frac{aq^2}{b^2};q^2\right)_\infty}{\left(\frac{aq}{b};q\right)_\infty\left(-\frac{q}{b};q\right)_\infty},
\tag{4}
$$

which is q-analog of Kummer's theorem for Gauss's hypergeometric function.

In 1973, Andrews [1] also derived the summation formula (4) by following a different method. In the same paper, he also derives the following summation formulas:

$$
{}_2\phi_2\left[\begin{array}{c} a,\ b, \\ \sqrt{qab},\ -\sqrt{qab}, \end{array};q,-q\right] = \frac{(-q;q)_\infty(aq;q^2)_\infty\left(bq;q^2\right)_\infty}{(qab;q^2)_\infty}
\tag{5}
$$

and

$$
{}_2\phi_2\left[\begin{array}{c} a,\ \frac{q}{a}, \\ b,\ -q, \end{array};q,-b\right] = \frac{(ab;q^2)_\infty\left(\frac{qb}{a};q^2\right)_\infty}{(b;q)_\infty},
\tag{6}
$$

which are q-analogues of Gauss's second summation theorem and Bailey's theorem. Andrews derive these formulas with the help of Jackson's identity [4] given below:

$$
{}_2\phi_2\left[\begin{array}{c} a,\ b, \\ c,\ az, \end{array};q,\frac{c}{b}z\right] = \frac{(z;q)_\infty}{(az;q)_\infty}{}_2\phi_1\left[\begin{array}{c} a,\ \frac{c}{b}, \\ c, \end{array};q,z\right].
\tag{7}
$$

In the theory of Gauss's hypergeometric function, Lavoie *et.al.* [7-9] derived many results closely related Kummer's theorem, Gauss's second summation theorem and Bailey's theorem. Recently Rakha *et.al.* [10] derived the following q-analogues of results closely related to Kummer's theorem:

$$
\begin{aligned}
&{}_2\phi_1\left[\begin{array}{c} a, bq, \\ \frac{aq}{b}, \end{array};q,-\frac{q}{b}\right] = \frac{b(-q;q)_\infty}{a(1-b)\left(\frac{aq}{b};q\right)_\infty\left(-\frac{1}{b};q\right)_\infty} \\
&\times\left\{(a;q^2)_\infty\left(\frac{aq}{b^2};q^2\right)_\infty - (aq;q^2)_\infty\left(\frac{a}{b^2};q^2\right)_\infty\right\},
\end{aligned}
\tag{8}
$$

$$
{}_2\phi_1\left[\begin{matrix} a, bq^2, \\ \frac{aq}{b}, \end{matrix}\ ;q,-\tfrac{q}{b}\ \right]=\frac{b^2(-q;q)_\infty}{a^2(1-b)(1-bq)\left(\frac{aq}{b};q\right)_\infty\left(-\frac{1}{bq};q\right)_\infty}
$$

$$
\times\left[\begin{matrix} (aq;q^2)_\infty\left(\frac{a}{b^2};q^2\right)_\infty\left\{q\left(1-\frac{a}{b^2q^2}\right)+(1-a)\right\} \\ -(a;q^2)_\infty\left(\frac{aq}{b^2};q^2\right)_\infty\left\{q\left(1-\frac{a}{b^2q^2}\right)+\left(1-\frac{a}{b^2}\right)\right\} \end{matrix}\right], \tag{9}
$$

$$
{}_2\phi_1\left[\begin{matrix} a, bq^3, \\ \frac{aq}{b}, \end{matrix}\ ;q,-\tfrac{q}{b}\ \right]
$$

$$
=\frac{b^3(-q;q)_\infty}{a^3(1-b)(1-bq)(1-bq^2)\left(\frac{aq}{b};q\right)_\infty\left(-\frac{1}{bq};q\right)_\infty}
$$

$$
\times\left[(a;q^2)_\infty\left(\frac{aq}{b^2};q^2\right)_\infty X_1-(aq;q^2)_\infty\left(\frac{aq^2}{b^2};q^2\right)_\infty Y_1\right], \tag{10}
$$

where

$$
X_1=q^2\left(1-\tfrac{a}{b^2q^4}\right)\left\{q\left(1-\tfrac{a}{b^2q^2}\right)+\left(1-\tfrac{a}{b^2}\right)\right\}
$$

$$
+\left(1-\tfrac{a}{b^2q}\right)\left\{q\left(1-\tfrac{a}{b^2q}\right)+(1-aq)\right\}
$$

and

$$
Y_1=q^2\left(1-\tfrac{a}{b^2}\right)\left(1-\tfrac{a}{b^2q^4}\right)\left\{q\left(1-\tfrac{a}{b^2q}\right)+(1-a)\right\}
$$

$$
+\left(1-\tfrac{a}{b^2q}\right)(1-a)\left\{q\left(1-\tfrac{a}{b^2q}\right)+\left(1-\tfrac{aq}{b^2}\right)\right\}
$$

$$
{}_2\phi_1\left[\begin{matrix} a, \frac{b}{q}, \\ \frac{aq}{b}, \end{matrix}\ ;q,-\tfrac{q}{b}\ \right]=\frac{(-q;q)_\infty}{\left(\frac{aq}{b};q\right)_\infty\left(-\frac{q}{b};q\right)_\infty}
$$

$$
\times\left\{(aq;q^2)_\infty\left(\frac{aq^2}{b^2};q^2\right)_\infty+(a;q^2)_\infty\left(\frac{aq^3}{b^2};q^2\right)_\infty\right\}, \tag{11}
$$

$$
{}_2\phi_1\left[\begin{matrix} a, \frac{b}{q^2}, \\ \frac{aq}{b}, \end{matrix}\ ;q,-\tfrac{q}{b}\ \right]=\frac{(-q;q)_\infty}{\left(\frac{aq}{b};q\right)_\infty\left(-\frac{q}{b};q\right)_\infty}
$$

$$
\times\left[\begin{matrix} (aq;q^2)_\infty\left(\frac{aq^4}{b^2};q^2\right)_\infty\left\{\left(1-\frac{aq^2}{b^2}\right)+\frac{(1-a)}{q}\right\} \\ +\frac{1}{q}(1+q)(a;q^2)_\infty\left(\frac{aq^3}{b^2};q^2\right)_\infty \end{matrix}\right], \tag{12}
$$

$$
{}_2\phi_1\left[\begin{matrix} a, \frac{b}{q^3}, \\ \frac{aq}{b}, \end{matrix}\ ;q,-\tfrac{q}{b}\ \right]=\frac{(-q;q)_\infty}{\left(\frac{aq}{b};q\right)_\infty\left(-\frac{q}{b};q\right)_\infty}
$$

$$
\times\left[(aq;q^2)_\infty\left(\frac{aq^4}{b^2};q^2\right)_\infty X_3+(a;q^2)_\infty\left(\frac{aq^5}{b^2};q^2\right)_\infty Y_3\right], \tag{13}
$$

where

$$
X_3=\left\{\left(1-\frac{aq^2}{b^2}\right)+\frac{(1-a)}{q}\right\}+\frac{1}{q^3}(1+q)(1-a),
$$

and

$$Y_3 = \frac{1}{q}(1+q)\left(1-\frac{aq^3}{b^2}\right)+\frac{1}{q^2}\left\{\left(1-\frac{aq^3}{b^2}\right)+\frac{(1-aq)}{q}\right\},$$

$$(14)\quad {}_2\phi_1\left[\begin{matrix}a,b\\ \frac{aq^2}{b}\end{matrix};q,-\frac{q}{b}\right]=\frac{(-q;q)_\infty}{\left(1-\frac{q}{b}\right)\left(\frac{aq^2}{b};q\right)_\infty\left(-\frac{q}{b};q\right)_\infty}\times\left\{(aq;q^2)_\infty\left(\frac{aq^2}{b^2};q^2\right)_\infty-\frac{q}{b}(a;q^2)_\infty\left(\frac{aq^3}{b^2};q^2\right)_\infty\right\},$$

$$(15)\quad {}_2\phi_1\left[\begin{matrix}a,b\\ \frac{aq^3}{b}\end{matrix};q,-\frac{q}{b}\right]=\frac{(-q;q)_\infty}{\left(1-\frac{q^2}{b}\right)\left(1-\frac{q}{b}\right)\left(\frac{aq^3}{b};q\right)_\infty\left(-\frac{q}{b};q\right)_\infty}\times\left\{\begin{matrix}(aq;q^2)_\infty\left(\frac{aq^2}{b^2};q^2\right)-\frac{q}{b}(1+q)(a;q^2)_\infty\left(\frac{aq^3}{b^2};q^2\right)_\infty\\ +\frac{q^3}{b^2}(1-a)(aq;q^2)_\infty\left(\frac{aq^4}{b^2};q^2\right)_\infty\end{matrix}\right\},$$

$$(16)\quad {}_2\phi_1\left[\begin{matrix}a,b,\\ \frac{a}{b},\end{matrix};q,-\frac{q}{b}\right]=\frac{(-q;q)_\infty}{\left(\frac{a}{b};q\right)_\infty\left(-\frac{1}{b};q\right)_\infty}\times\left\{(aq;q^2)_\infty\left(\frac{a}{b^2};q^2\right)_\infty+\frac{1}{b}(a;q^2)_\infty\left(\frac{aq}{b^2};q^2\right)_\infty\right\}$$

and

$$(17)\quad {}_2\phi_1\left[\begin{matrix}a,b,\\ \frac{a}{bq},\end{matrix};q,-\frac{q}{b}\right]=\frac{(-q;q)_\infty}{\left(\frac{a}{bq};q\right)_\infty\left(-\frac{1}{bq};q\right)_\infty}\times\left\{\begin{matrix}\left(1-\frac{a}{b^2q}\right)(aq;q^2)_\infty\left(\frac{a}{b^2};q^2\right)_\infty+\frac{1}{bq}(1+q)(a;q^2)_\infty\left(\frac{a}{b^2q};q^2\right)_\infty\\ +\frac{1}{b^2q}\left(\frac{a}{q};q^2\right)_\infty\left(\frac{a}{b^2};q^2\right)_\infty\end{matrix}\right\}.$$

The main objective of this paper is to derive the q-analogues of results closely related to Gauss's second summation theorem and Bailey's theorem. Applications and special cases are also derived.

2. Main results

In this section, firstly we derive the following results, which are basic analogues of results closely related to the Gauss's second summation theorem:

$$(18)\qquad {}_2\phi_2\left[\begin{matrix} a, & b \\ q\sqrt{ab} & -q\sqrt{ab} \end{matrix} : q; -q\right] = \frac{(-q;q)_\infty}{(b-a)(q^2ab;q^2)_\infty} \times \left\{b(a;q^2)_\infty\left(bq;q^2\right)_\infty - a(aq;q^2)_\infty\left(b;q^2\right)_\infty\right\},$$

$$(19)\qquad {}_2\phi_2\left[\begin{matrix} a, b, \\ q\sqrt{ab}, -q\sqrt{ab}, \end{matrix} ; q, -q^2\right] = \frac{(-q;q)_\infty}{(b-a)(q^2ab;q^2)_\infty} \times \left\{(a;q^2)_\infty\left(bq;q^2\right)_\infty - (aq;q^2)_\infty\left(b;q^2\right)_\infty\right\},$$

$$(20)\qquad {}_2\phi_2\left[\begin{matrix} a, & b \\ q^{\frac{3}{2}}\sqrt{ab} & -q^{\frac{3}{2}}\sqrt{ab} \end{matrix} : q; -q\right] = \frac{(-q;q)_\infty}{(a-b)(q^3ab;q^2)_\infty} \times \left[\begin{matrix} \left\{\frac{abq(1+q)(b-a)}{(b-aq)}\right\}(a;q^2)_\infty\left(b;q^2\right)_\infty \\ -\left\{\frac{b^2(1-a)}{(b-aa)} + \frac{a^2(1-b)}{(bq-a)}\right\}(aq;q^2)_\infty\left(bq;q^2\right)_\infty \end{matrix}\right],$$

$$(21)\qquad {}_2\phi_2\left[\begin{matrix} a, b, \\ q^{\frac{3}{2}}\sqrt{ab}, -q^{\frac{3}{2}}\sqrt{ab}, \end{matrix} ; q, -q^3\right] = \frac{(-q;q)_\infty}{(bq-a)(b-aq)(q^3ab;q^2)_\infty} \times \left[\begin{matrix} (aq;q^2)_\infty\left(bq;q^2\right)_\infty(q-b+1-a) \\ -(1+q)(a;q^2)_\infty\left(b;q^2\right)_\infty \end{matrix}\right],$$

$$(22)\qquad {}_2\phi_2\left[\begin{matrix} a, b, \\ q^2\sqrt{ab}, -q^2\sqrt{ab}, \end{matrix} ; q, -q^4\right] = \frac{(-q;q)_\infty}{q(q^2b-a)(b-a)(b-q^2a)(q^4ab;q^2)_\infty} \times \left\{(a;q^2)_\infty\left(bq;q^2\right)_\infty {X_1}'^{-(aq;q^2)_\infty\left(bq^2;q^2\right)_\infty {Y_1}'}\right\},$$

where

$${X_1}' = \left[(q^2-b)(1+q) + (q-bq^2+1-aq)\right]$$

and

$${Y_1}' = \left[(q^2-b)\left\{q(1-b)+(1-a)\right\} + (1-a)(q-1)(1-bq)\right]$$

$$(23)\qquad {}_2\phi_2\left[\begin{matrix} a, b, \\ \sqrt{ab}, -\sqrt{ab}, \end{matrix} ; q, -1\right] = \frac{(-q;q)_\infty}{(ab;q^2)_\infty} \times \left\{(aq;q^2)_\infty\left(b;q^2\right)_\infty + (a;q^2)_\infty\left(bq;q^2\right)_\infty\right\},$$

(24)
$$ {}_2\phi_2\left[\begin{array}{c} a, b, \\ \sqrt{\frac{ab}{q}}, -\sqrt{\frac{ab}{q}}, \end{array}; q, -\frac{1}{q}\right] = \frac{(-q;q)_\infty}{q\left(\frac{ab}{q};q^2\right)_\infty} \times \left\{(aq;q^2)_\infty \left(bq;q^2\right)_\infty (q-b+a-1) + (1+q)(a;q^2)_\infty \left(b;q^2\right)_\infty\right\}, $$

(25)
$$ {}_2\phi_2\left[\begin{array}{c} a, b, \\ \frac{1}{q}\sqrt{ab}, -\frac{1}{q}\sqrt{ab}, \end{array}; q, -\frac{1}{q^2}\right] = \frac{(-q;q)_\infty}{q\left(\frac{ab}{q};q^2\right)_\infty} \times \left\{(aq;q^2)_\infty \left(b;q^2\right)_\infty {X_2}' + (a;q^2)_\infty \left(bq;q^2\right)_\infty {Y_2}'\right\}, $$

where
$$ {X_2}' = \left\{\left(1-\frac{b}{q^2}\right) + \frac{(1-a)}{q}\right\} + \frac{1}{q^3}(1+q)(1-a) $$

and
$$ {Y_2}' = \frac{1}{q^2}\left\{\left(1-\frac{b}{q^2}\right) + \frac{(1-a)}{q}\right\} + \frac{1}{q}(1+q)\left(1-\frac{b}{q}\right). $$

Next, we derive the following basic analogues of the results closely related to the Bailey's theorem:

(26)
$$ {}_2\phi_2\left[\begin{array}{c} a, \frac{q^2}{a}, \\ b, -q, \end{array}; q, -\frac{b}{q}\right] = \frac{1}{\left(1-\frac{q}{a}\right)(b;q)_\infty} \times \left\{\left(\frac{ab}{q};q^2\right)_\infty \left(\frac{b}{a};q^2\right)_\infty - \frac{q}{a}\left(\frac{ab}{q^2};q^2\right)_\infty \left(\frac{bq}{a};q^2\right)_\infty\right\}, $$

(27)
$$ {}_2\phi_2\left[\begin{array}{c} a, \frac{q^3}{a}, \\ b, -q, \end{array}; q, -\frac{b}{q^2}\right] = \frac{1}{\left(1-\frac{q^2}{a}\right)\left(1-\frac{q}{a}\right)(b;q)_\infty} \times \left\{\begin{array}{c} \left(\frac{ab}{q^2};q^2\right)_\infty \left(\frac{b}{aq};q^2\right)_\infty - \frac{q}{a}(1+q)\left(\frac{ab}{q^3};q^2\right)_\infty \left(\frac{b}{a};q^2\right)_\infty \\ + \frac{q^3}{a^2}\left(1-\frac{ab}{q^3}\right)\left(\frac{ab}{q^2};q^2\right)_\infty \left(\frac{bq}{a};q^2\right)_\infty \end{array}\right\}, $$

(28)
$$ {}_2\phi_2\left[\begin{array}{c} a, \frac{1}{a}, \\ b, -q, \end{array}; q, -bq\right] = \frac{a}{(1+a)(b;q)_\infty} \times \left\{\left(abq;q^2\right)_\infty \left(\frac{b}{a};q^2\right)_\infty + \frac{1}{a}\left(ab;q^2\right)_\infty \left(\frac{bq}{a};q^2\right)_\infty\right\}, $$

and

(29)

$$ {}_2\phi_2\left[\begin{matrix} a, \frac{1}{aq}, \\ b, -q, \end{matrix}; q, -bq^2\right] = \frac{a^2q}{(1+aq)(1+a)(b;q)_\infty} \times \left\{ \begin{matrix} \left(1-\frac{b}{a}\right)\left(abq^2;q^2\right)_\infty \left(\frac{bq}{a};q^2\right)_\infty + \frac{1}{aq}(1+q)\left(abq;q^2\right)_\infty \left(\frac{b}{a};q^2\right)_\infty \\ + \frac{1}{a^2q}\left(ab;q^2\right)_\infty \left(\frac{bq}{a};q^2\right)_\infty \end{matrix} \right\}. $$

3. Derivations of the main results

In order to prove the result (18), we denote the left hand side of (18) as ϕ, then

$$ \phi = {}_2\phi_2\left[\begin{matrix} a, & b \\ q\sqrt{ab} & -q\sqrt{ab} \end{matrix}; q, -q\right] $$

$$ = \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(q;q)_n \left(q\sqrt{ab};q\right)_n \left(-q\sqrt{ab};q\right)_n} z^n $$

$$ = \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(q;q)_n \left(q\sqrt{ab};q\right)_n \left(-q\sqrt{ab};q\right)_n} z^n \frac{(b-a)}{(b-a)} $$

$$ = \frac{1}{(b-a)} \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(q;q)_n \left(q\sqrt{ab};q\right)_n \left(-q\sqrt{ab};q\right)_n} z^n \left(b - a + abq^n - abq^n\right) $$

$$ = \frac{1}{(b-a)} \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(q;q)_n \left(q\sqrt{ab};q\right)_n \left(-q\sqrt{ab};q\right)_n} z^n \left\{ \begin{matrix} b\left(1-aq^n\right) \\ -a\left(1-bq^n\right) \end{matrix} \right\}. $$

Hence

$$ \phi = \frac{b}{(b-a)} \sum_{n=0}^{\infty} \frac{(a;q)_n \left(1-aq^n\right) (b;q)_n}{(q;q)_n \left(q\sqrt{ab};q\right)_n \left(-q\sqrt{ab};q\right)_n} z^n $$

$$ - \frac{a}{(b-a)} \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n \left(1-bq^n\right)}{(q;q)_n \left(q\sqrt{ab};q\right)_n \left(-q\sqrt{ab};q\right)_n} z^n. $$

On using the known result, namely

$$(\alpha)_n\,(1-\alpha q^n) = (\alpha)_{n+1} = (1-\alpha)\,(\alpha q)_n,$$

we can write

$$\phi = \frac{b(1-a)}{(b-a)} \sum_{n=0}^{\infty} \frac{(aq;q)_n (b;q)_n}{(q;q)_n \Big(q\sqrt{ab};q\Big)_n \Big(-q\sqrt{ab};q\Big)_n}\; z^n$$

$$-\frac{a(1-b)}{(b-a)} \sum_{n=0}^{\infty} \frac{(a;q)_n (bq;q)_n}{(q;q)_n \Big(q\sqrt{ab};q\Big)_n \Big(-q\sqrt{ab};q\Big)_n}\; z^n.$$

Or

$$\begin{array}{c} \phi = \frac{b(1-a)}{(b-a)}\ {}_2\phi_2 \left[\begin{array}{cc} aq, & b \\ q\sqrt{ab} & -q\sqrt{ab} \end{array} : q; -q \right] \\ -\frac{a(1-b)}{(b-a)}\ {}_2\phi_2 \left[\begin{array}{cc} a, & bq \\ q\sqrt{ab} & -q\sqrt{ab} \end{array} : q; -q \right] \end{array},$$

which implies that

$$\begin{array}{c} {}_2\phi_2 \left[\begin{array}{cc} a, & b \\ q\sqrt{ab} & -q\sqrt{ab} \end{array} : q; -q \right] = \\ \frac{b(1-a)}{(b-a)}\ {}_2\phi_2 \left[\begin{array}{cc} aq, & b \\ q\sqrt{ab} & -q\sqrt{ab} \end{array} : q; -q \right] \\ -\frac{a(1-b)}{(b-a)}\ {}_2\phi_2 \left[\begin{array}{cc} a, & bq \\ q\sqrt{ab} & -q\sqrt{ab} \end{array} : q; -q \right]. \end{array} \tag{30}$$

Now, the right hand side of (30) can be evaluated with the help of summation formula (5) by replacing a to aq or b to bq and also by permitting the use of identity $(1-a)(aq^2;q^2)_\infty = (a;q^2)_\infty$, then after little simplification, we obtain

$$\begin{array}{c} {}_2\phi_2 \left[\begin{array}{cc} a, & b \\ q\sqrt{ab} & -q\sqrt{ab} \end{array} : q; -q \right] = \frac{(-q;q)_\infty}{(b-a)(q^2ab;q^2)_\infty} \\ \times \left\{ b(a;q^2)_\infty \big(bq;q^2\big)_\infty - a(aq;q^2)_\infty \big(b;q^2\big)_\infty \right\}. \end{array} \tag{31}$$

This completes the proof of result (18).
In similar manner, we can easily prove the result (20), so we omit the details.

Again, in order to derive (19), we put $c = q\sqrt{ab}$ and $z = -q\sqrt{\frac{b}{a}}$ in (6), then after some rearrangement, we get

$$
\begin{aligned}
&{}_2\phi_2\left[\begin{matrix} a, b, \\ q\sqrt{ab}, -q\sqrt{ab}, \end{matrix}\ ; q, q^2\right] = \\
&\frac{\left(-q\sqrt{\frac{b}{a}};q\right)_\infty}{(-q\sqrt{ab};q)_\infty}\,{}_2\phi_1\left[\begin{matrix} a, q\sqrt{\frac{a}{b}}, \\ q\sqrt{ab}, \end{matrix}\ ; q, -q\sqrt{\frac{b}{a}}\right]
\end{aligned} \tag{32}
$$

On the other hand, if we replace $b = \sqrt{\frac{a}{b}}$ in (2), we obtain

$$
\begin{aligned}
{}_2\phi_1\left[\begin{matrix} a, q\sqrt{\frac{a}{b}}, \\ q\sqrt{ab}, \end{matrix}\ ; q, -q\sqrt{\frac{b}{a}}\right] = \ & \frac{(-q;q)_\infty}{\sqrt{a}\left(\sqrt{b}-\sqrt{a}\right)\left(q\sqrt{ab};q\right)_\infty\left(-\sqrt{\frac{b}{a}};q\right)_\infty} \\
& \times\left\{(a;q^2)_\infty\left(bq;q^2\right)_\infty - (aq;q^2)_\infty\left(b;q^2\right)_\infty\right\}.
\end{aligned} \tag{33}
$$

On noting

$$
\frac{1}{\sqrt{a}\left(\sqrt{b}-\sqrt{a}\right)\left(-\sqrt{\frac{b}{a}};q\right)_\infty} = \frac{1}{(b-a)\left(-q\sqrt{\frac{b}{a}};q\right)_\infty},
$$

relation (31) becomes

$$
\begin{aligned}
{}_2\phi_1\left[\begin{matrix} a, q\sqrt{\frac{a}{b}}, \\ q\sqrt{ab}, \end{matrix}\ ; q, -q\sqrt{\frac{b}{a}}\right] = \ & \frac{(-q;q)_\infty}{(b-a)\left(q\sqrt{ab};q\right)_\infty\left(-q\sqrt{\frac{b}{a}};q\right)_\infty} \\
& \times\left\{(a;q^2)_\infty\left(bq;q^2\right)_\infty - (aq;q^2)_\infty\left(b;q^2\right)_\infty\right\}.
\end{aligned} \tag{34}
$$

Now, on using the above result in the right hand side of (30) and then after little simplification, we arrive at

$$
\begin{aligned}
{}_2\phi_2\left[\begin{matrix} a, b, \\ q\sqrt{ab}, -q\sqrt{ab}, \end{matrix}\ ; q, -q^2\right] = \ & \frac{(-q;q)_\infty}{(b-a)(q^2ab;q^2)_\infty} \\
& \times\left\{(a;q^2)_\infty\left(bq;q^2\right)_\infty - (aq;q^2)_\infty\left(b;q^2\right)_\infty\right\}.
\end{aligned}
$$

This completes the proof of (19).

Following the similar manner, we can easily derive the remaining results. The following table shows how we could get the different results by substituting different values in Jackson's theorem and results required.

S. No.	Derivation of	Values in Jackson's theorem	Use of the result
1	(21)	$c = q^{\frac{3}{2}}\sqrt{ab}$,$z = -q^{\frac{3}{2}}\sqrt{\frac{b}{a}}$	(9)
2	(22)	$c = q^{2}\sqrt{ab}$,$z = -q^{2}\sqrt{\frac{b}{a}}$	(10)
3	(23)	$c = \sqrt{ab}$,$z = -\sqrt{\frac{b}{a}}$	(11)
4	(24)	$c = \sqrt{\frac{ab}{q}}$,$z = -\sqrt{\frac{b}{aq}}$	(12)
5	(25)	$c = \frac{1}{q}\sqrt{ab}$,$z = -\frac{1}{q}\sqrt{\frac{b}{a}}$	(13)
6	(26)	$c = b$,$b = \frac{q^{2}}{a}$ and $z = -\frac{q}{a}$	(14)
7	(27)	$c = b$,$b = \frac{q^{3}}{a}$ and $z = -\frac{q}{a}$	(15)
8	(28)	$c = b$,$b = \frac{1}{q}$ and $z = -\frac{q}{a}$	(16)
9	(29)	$c = b$,$b = \frac{1}{aq}$ and $z = -\frac{q}{a}$	(17)

For the shake of brevity we omit the detailed proof.

4. Applications

In this section, we consider some consequences of the main results derived in the preceding section. To illustrate this, we deduce known result (14) by using the results (18) and (19). For this, if we put $a = A$, $b = \frac{Aq^{2}}{B^{2}}$, $c = \frac{Aq^{2}}{B}$ and $z = -\frac{q}{B}$ in the Jackson's identity, we obtain

$$
{}_2\phi_2\left[\begin{matrix} A, & \frac{Aq^{2}}{B^{2}} \\ \frac{Aq^{2}}{B} & A\left(-\frac{q}{B}\right) \end{matrix}; q, B.\left(-\frac{q}{B}\right)\right] = \frac{\left(-\frac{q}{B};q\right)_{\infty}}{\left\{A.\left(-\frac{q}{B}\right);q\right\}_{\infty}} \times {}_2\phi_1\left[\begin{matrix} A, & \frac{Aq^{2}}{B}\Big/\frac{Aq^{2}}{B^{2}} \\ & \frac{Aq^{2}}{B} \end{matrix}; q, \left(-\frac{q}{B}\right)\right],
$$

which can be written as

$$
{}_2\phi_1\left[\begin{matrix} A, & B \\ \frac{Aq^{2}}{B} & \end{matrix}; q, -\frac{q}{B}\right] = \frac{\left(-\frac{Aq}{B};q\right)_{\infty}}{\left(-\frac{q}{B};q\right)_{\infty}} {}_2\phi_2\left[\begin{matrix} A, & \frac{Aq^{2}}{B^{2}} \\ \frac{Aq^{2}}{B} & -\frac{Aq}{B} \end{matrix}; q, -q\right].
$$

Now on writing the right had side of above equation as I, and on expanding the series involved, we get

$$
I = \frac{\left(-\frac{Aq}{B};q\right)_{\infty}}{\left(-\frac{q}{B};q\right)_{\infty}} \sum_{n=0}^{\infty} \frac{(A;q)_n \left(\frac{Aq^{2}}{B^{2}};q\right)_n}{(q;q)_n \left(\frac{Aq^{2}}{B};q\right)_n \left(-\frac{Aq}{B};q\right)_n} \left\{(-1)^n q^{\binom{n}{2}}\right\}^{1+2-2} (-q)^n
$$

$$= \frac{\left(-\frac{Aq}{B};q\right)_\infty}{\left(-\frac{q}{B};q\right)_\infty} \sum_{n=0}^{\infty} \frac{(A;q)_n \left(\frac{Aq^2}{B^2};q\right)_n \left(1-\frac{Aq^{n+1}}{B}\right)}{(q;q)_n \left(\frac{Aq^2}{B};q\right)_n \left(1-\frac{Aq}{B}\right)\left(-\frac{Aq^2}{B};q\right)_{n-1}\left(1-\frac{Aq^{n+1}}{B}\right)} \times \left\{(-1)^n q^{\binom{n}{2}}\right\}(-q)^n .$$

This can also be written as:

$$I = \frac{\left(-\frac{Aq}{B};q\right)_\infty}{\left(-\frac{q}{B};q\right)_\infty} \frac{1}{\left(1+\frac{Aq}{B}\right)} \times \left[\sum_{n=0}^{\infty} \frac{(A;q)_n \left(\frac{Aq^2}{B^2};q\right)_n \left(1+\frac{Aq^{n+1}}{B}\right)}{(q;q)_n \left(\frac{Aq^2}{B};q\right)_n \left(-\frac{Aq^2}{B};q\right)_n} \left\{(-1)^n q^{\binom{n}{2}}\right\}(-q)^n\right]$$

$\Rightarrow$

$$I = \frac{\left(-\frac{Aq}{B};q\right)_\infty}{\left(1+\frac{Aq}{B}\right)\left(-\frac{q}{B};q\right)_\infty} \left[\sum_{n=0}^{\infty} \frac{(A;q)_n \left(\frac{Aq^2}{B^2};q\right)_n \left\{(-1)^n q^{\binom{n}{2}}\right\}(-q)^n}{(q;q)_n \left(\frac{Aq^2}{B};q\right)_n \left(-\frac{Aq^2}{B};q\right)_n} + \left(\frac{Aq}{B}\right) \sum_{n=0}^{\infty} \frac{(A;q)_n \left(\frac{Aq^2}{B^2};q\right)_n \left\{(-1)^n q^{\binom{n}{2}}\right\}(-q)^{n+n}}{(q;q)_n \left(\frac{Aq^2}{B};q\right)_n \left(-\frac{Aq^2}{B};q\right)_n}\right] .$$

Hence, we obtain

$${}_2\phi_1\left[\begin{matrix} A, & B \\ \frac{Aq^2}{B} & \end{matrix}; q, -\frac{q}{B}\right] = \frac{\left(-\frac{Aq}{B};q\right)_\infty}{\left(1+\frac{Aq}{B}\right)\left(-\frac{q}{B};q\right)_\infty} \times \left\{{}_2\phi_2\left[\begin{matrix} A, & \frac{Aq^2}{B^2} \\ \frac{Aq^2}{B} & -\frac{Aq^2}{B} \end{matrix}; q, -q\right] + \frac{Aq}{B}\,{}_2\phi_2\left[\begin{matrix} A, & \frac{Aq^2}{B^2} \\ \frac{Aq^2}{B} & -\frac{Aq^2}{B} \end{matrix}; q, -q^2\right]\right\}$$

It is interesting to observe that, on using (18) and (19), and after certain simplification the above result reduces to (14).
In the same manner, using (20) and (21), we can also derive (15).

5. Special cases

It is interesting to observe that in view of the following limit formulas:

$$\lim_{q\to 1^-} \Gamma_q(\alpha) = \Gamma(\alpha)$$

and

$$\underset{q\to 1^-}{Lim}\,\frac{(q^{\alpha};q)_n}{(1-q)^n}=(\alpha)_n$$

the main results (18) and (19) provides, the q-extensions of the following result:

$$(35)\qquad {}_2F_1\left[\begin{array}{c} a,b \\ \frac{1}{2}(a+b+2) \end{array};\frac{1}{2}\right]=\Gamma\left(\tfrac{1}{2}\right)\Gamma\left(\tfrac{1}{2}a+\tfrac{1}{2}b+1\right)\times\left\{\frac{1}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}-\frac{1}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b\right)}\right\}.$$

Again, if we take limit $q\to 1^-$ of results (20) and (21), we obtain the following:

$$(36)\qquad {}_2F_1\left[\begin{array}{c} a,b \\ \frac{1}{2}(a+b+3) \end{array};\frac{1}{2}\right]=\Gamma\left(\tfrac{1}{2}\right)\Gamma\left(\tfrac{1}{2}a+\tfrac{1}{2}b+\tfrac{3}{2}\right)\times\left\{\frac{\frac{1}{2}(b+a-1)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}-\frac{2}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b-\frac{1}{2}\right)}\right\}.$$

Finally, it is interesting to note that, the results (18) and (19) are q-analogues of the same result (35), and (20) and (21) are q-analogues of the result (36). Moreover, our results shows that, it is possible that any hypergeometric identity bears more than one basic analogue.

Acknowledgments

The authors would like to express their appreciation to the referees for their valuable suggestions which helped to better presentation of this paper.

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