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On a Summation due to Ramanujan and others and their generalizations

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Abstract. The aim of this short note is to demonstrate how one can obtain three general summations by employing generalized Kummer's summation theorem obtained earlier by Choi. As special cases, we mention a large number of summations including summations due to Ramanujan and Brychkov. The summations given in this note are simple, interesting, easily established and may be useful.

1 Introduction

We start with the following interesting summations,

$$1-\left(\frac{1}{2}\right)^2+\left(\frac{1.3}{2.4}\right)^2-\ldots = \frac{\sqrt{\pi}}{\sqrt{2}\Gamma^2(\frac{3}{4})} \tag{1.1}$$

$$1-\left(\frac{1}{4}\right)^2+\left(\frac{1.5}{4.8}\right)^2-\ldots = \frac{\Gamma(\frac{1}{8})\Gamma(\frac{3}{8})}{2^{\frac{7}{4}}\pi^{\frac{3}{2}}} \tag{1.2}$$

$$1-\left(\frac{1}{3}\right)+\left(\frac{1}{5}\right)-\ldots = \frac{\pi}{4} \tag{1.3}$$

The first summation (1.1) is due to Ramanujan[1], the second summation (1.2) is due to Brychkov [2] while (1.3) is the well known summation due to Leibnitz obtained in 1673[4].

The above summations can be proved quite easily by using the classical Kummer's summation theorem [5].

$${}_2F_1\left[\begin{matrix} a, b \\ 1+a-b \end{matrix} ; -1\right] = \frac{\Gamma(\frac{1}{2}a+1)\Gamma(1+a-b)}{\Gamma(a+1)\Gamma(1+\frac{1}{2}a-b)} \tag{1.4}$$

by taking

(i) $a=b=\frac{1}{2}$

(ii) $a=b=\frac{1}{4}$

(iii) $a=1$ and $b=\frac{1}{2}$

respectively.

Recently, Choi[3] established the generalization of the classical Kummer's summation theorem(1.4) and obtained nineteen results in the form of a single result which is given by

$$
{}_2F_1\left[\begin{matrix} a, b \\ 1+a-b+i \end{matrix} ; -1\right]
= \frac{\Gamma(\frac{1}{2})\Gamma(1+a-b+i)\Gamma(1-b)}{2^a\Gamma(1-b+\frac{1}{2}(i+\mid i\mid))} \cdot \left[\frac{A_i}{\Gamma(\frac{1}{2}a+\frac{1}{2}i+\frac{1}{2}-[\frac{1+i}{2}])\Gamma(1+\frac{1}{2}a-b+\frac{1}{2}i)} + \frac{B_i}{\Gamma(\frac{1}{2}a+\frac{1}{2}i-[\frac{i}{2}])\Gamma(\frac{1}{2}+\frac{1}{2}a-b+\frac{1}{2}i)}\right] \quad (1.5)
$$

for $i = 0, \pm 1, \pm 2, ..., \pm 9$.

Here $[x]$ is the greatest integer less than or equal to x and its modulus is denoted by $\mid x \mid$. The coefficients A_i and B_i are given in the following tables.

Table 1

i	A_i	B_i
0	1	0
1	-1	1
2	$1+a-b$	-2
3	$3a-2b-5$	$2a-b+1$
4	$2a^2-4ab+b^2+8a-3b+2$	$4(b-a-2)$
5	$10ab-4a^2-5b^2-26a+25b-32$	$4a^2-6ab+b^2+14a-3b+2$
6	$4a^3-12a^2b+9ab^2-b^3+36a^2-51ab+6b^2+74a-11b+6$	$16ab-8a^2-6b^2-48a+34b-52$
7	$7b^3-28ab^2+28a^2b-8a^3-100a^2+196ab-70b^2-352a+245b-302$	$8a^3-20a^2b+12ab^2-b^3+68a^2-76ab+6b^2+128a-11b+6$
8	$8a^4-32a^3b+40a^2b^2-16ab^3+b^4+128a^3-312a^2b+176ab^2-10b^3+624a^2-672ab+35b^2+896a-50b+24$	$8b^3-40ab^2+48a^2b-16a^3-192a^2+312ab-88b^2-640a+352b-512$
9	$-16a^4+72a^3b-108a^2b^2+60ab^3-9b^4-328a^3+972a^2b-792ab^2+150b^3-2240a^2+3612ab-999b^2-5696a+3162b-3984$	$16a^4-56a^3b+60a^2b^2-20ab^3+b^4+248a^3-516a^2b+240ab^2-10b^3+1160a^2-1028ab+35b^2+1576a-50b+24$

Table 2

i	A_i	B_i
-1	1	1
-2	$a-b-1$	2
-3	$2a-3b-4$	$2a-b-2$
-4	$2a^2-4ab+b^2-8a+5b+6$	$4(a-b-2)$
-5	$4a^2-10ab+5b^2-24a+25b+32$	$4a^2-6ab+b^2-16a+7b+12$
-6	$4a^3-12a^2b+9ab^2-b^3-36a^2+57ab-12b^2+92a-47b-60$	$8a^2-16ab+6b^2-48a+38b+64$
-7	$8a^3-28a^2b+28ab^2-7b^3-96a^2+196ab-77b^2+352a-294b-384$	$8a^3-20a^2b+12ab^2-b^3-72a^2+92ab-15b^2+184a-74b-120$
-8	$8a^4-32a^3b+40a^2b^2-16ab^3+b^4-128a^3+328a^2b-208ab^2+22b^3+688a^2-928ab+179b^2-1408a+638b+840$	$16a^3-48a^2b+40ab^2-8b^3-192a^2+328ab-104b^2+704a-408b-768$
-9	$16a^4-72a^3+108a^2b^2-60ab^3+9b^4-320a^3+972a^2b-828ab^2+174b^3+2240a^2-3936ab+1323b^2-6400a+4614b+6144$	$16a^4-56a^3b+60a^2b^2-20ab^3+b^4-256a^3+564a^2b-3000ab^2+26b^3+1376a^2-1568ab+251b^2-2816a+1066b+1680$

The aim of this short research paper is to provide generalizations of the summations (1.1),(1.2) and (1.3). As special cases, we obtained in all 27 results closely related to (1.1) to (1.3).

2 MAIN RESULTS

In this section, the following summations will be established.

$$1-\left(\frac{1}{2}\right)^2\frac{1}{(1+i)}+\left(\frac{1.3}{2.2}\right)^2\frac{1}{(1+i)(2+i)}-\dots$$
$$=\frac{\Gamma^2\left(\frac{1}{2}\right)\Gamma(1+i)}{\sqrt{2}\Gamma\left(\frac{1}{2}+\frac{1}{2}(i+\mid i\mid)\right)}\left[\frac{A_i}{\Gamma\left(\frac{3}{4}+\frac{1}{2}i-\left[\frac{1+i}{2}\right]\right)\Gamma\left(\frac{3}{4}+\frac{1}{2}i\right)}\right.$$
$$\left.+\frac{B_i}{\Gamma\left(\frac{1}{4}+\frac{1}{2}i-\left[\frac{i}{2}\right]\right)\Gamma\left(\frac{1}{4}+\frac{1}{2}i\right)}\right] \tag{2.1}$$

for $i=0,1,2,...,9$.

$$1-\left(\frac{1}{4}\right)^2\frac{1}{(1+i)}+\left(\frac{1.5}{4.4}\right)^2\frac{1}{(1+i)(2+i)}-\dots$$
$$=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(1+i)\Gamma\left(\frac{3}{4}\right)}{2^{\frac{1}{4}}\Gamma\left(\frac{3}{4}+\frac{1}{2}(i+\mid i\mid)\right)}\left[\frac{A_i}{\Gamma\left(\frac{5}{8}+\frac{1}{2}i-\left[\frac{1+i}{2}\right]\right)\Gamma\left(\frac{7}{8}+\frac{1}{2}i\right)}\right.$$
$$\left.+\frac{B_i}{\Gamma\left(\frac{1}{8}+\frac{1}{2}i-\left[\frac{i}{2}\right]\right)\Gamma\left(\frac{3}{8}+\frac{1}{2}i\right)}\right] \tag{2.2}$$

for $i=0,1,2,...,9$.

$$1-\frac{1}{2}\frac{1}{(\frac{3}{2}+i)}+\frac{1.3}{2.2}\frac{1}{(\frac{3}{2}+i)(\frac{5}{2}+i)}-\dots$$
$$=\frac{\Gamma^2\left(\frac{1}{2}\right)\Gamma(\frac{3}{2}+i)}{2\Gamma\left(\frac{1}{2}+\frac{1}{2}(i+\mid i\mid)\right)}\left[\frac{A_i}{\Gamma\left(1+\frac{1}{2}i-\left[\frac{1+i}{2}\right]\right)\Gamma\left(1+\frac{1}{2}i\right)}\right.$$
$$\left.+\frac{B_i}{\Gamma\left(\frac{1}{2}+\frac{1}{2}i-\left[\frac{i}{2}\right]\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}i\right)}\right] \tag{2.3}$$

for $i=0,1,2,...,9$.

Proof. The derivations of our general summations are quite straight-forward. For this, in (1.5), if we set

(i) $a=b=\frac{1}{2}$

(ii) $a=b=\frac{1}{4}$ and

(iii) $a=1$ and $b=\frac{1}{2}$

and after little simplification, we easily arrive at (2.1), (2.2) and (2.3).
This completes the proof of summations (2.1), (2.2) and (2.3). □

3 SPECIAL CASES

In this section, we shall mention, some of very interesting summations.

(i) **SPECIAL CASES OF (2.1)**

In (2.1), if we take $i = 0, 1, 2, ..., 9$, we get the following interesting results.

$$1-\left(\frac{1}{2}\right)^2+\left(\frac{1.3}{2.4}\right)^2-... = \frac{\sqrt{\pi}}{\sqrt{2}\Gamma^2\left(\frac{3}{4}\right)} \tag{3.1}$$

$$1-\left(\frac{1}{2}\right)^2\frac{1}{2}+\left(\frac{1.3}{2.4}\right)^2\frac{1}{3}-... = \frac{\sqrt{2\pi}}{\Gamma^2\left(\frac{3}{4}\right)}-\frac{2\sqrt{2}\Gamma^2\left(\frac{3}{4}\right)}{\pi^{\frac{3}{2}}} \tag{3.2}$$

$$1-\left(\frac{1}{2}\right)^2\frac{1}{3}+\left(\frac{1.3}{2.4}\right)^2\frac{1}{6}-... = \frac{16\sqrt{2\pi}}{9\Gamma^2\left(\frac{3}{4}\right)}-\frac{16\sqrt{2}\Gamma^2\left(\frac{3}{4}\right)}{3\pi^{\frac{3}{2}}} \tag{3.3}$$

$$1-\left(\frac{1}{2}\right)^2\frac{1}{4}+\left(\frac{1.3}{2.4}\right)^2\frac{1}{10}-... = \frac{16\sqrt{2\pi}}{5\Gamma^2\left(\frac{3}{4}\right)}-\frac{288\sqrt{2}\Gamma^2\left(\frac{3}{4}\right)}{25\pi^{\frac{3}{2}}} \tag{3.4}$$

$$1-\left(\frac{1}{2}\right)^2\frac{1}{5}+\left(\frac{1.3}{2.4}\right)^2\frac{1}{15}-... = \frac{4352\sqrt{2\pi}}{735\Gamma^2\left(\frac{3}{4}\right)}-\frac{4096\sqrt{2}\Gamma^2\left(\frac{3}{4}\right)}{175\pi^{\frac{3}{2}}} \tag{3.5}$$

$$1-\left(\frac{1}{2}\right)^2\frac{1}{6}+\left(\frac{1.3}{2.4}\right)^2\frac{1}{21}-... = \frac{14848\sqrt{2\pi}}{1323\Gamma^2\left(\frac{3}{4}\right)}-\frac{44032\sqrt{2}\Gamma^2\left(\frac{3}{4}\right)}{945\pi^{\frac{3}{2}}} \tag{3.6}$$

$$1-\left(\frac{1}{2}\right)^2\frac{1}{7}+\left(\frac{1.3}{2.4}\right)^2\frac{1}{28}-... = \frac{385024\sqrt{2\pi}}{17787\Gamma^2\left(\frac{3}{4}\right)}-\frac{106496\sqrt{2}\Gamma^2\left(\frac{3}{4}\right)}{1155\pi^{\frac{3}{2}}} \tag{3.7}$$

$$1-\left(\frac{1}{2}\right)^2\frac{1}{8}+\left(\frac{1.3}{2.4}\right)^2\frac{1}{36}-... = \frac{598016\sqrt{2\pi}}{14157\Gamma^2\left(\frac{3}{4}\right)}-\frac{15253504\sqrt{2}\Gamma^2\left(\frac{3}{4}\right)}{83655\pi^{\frac{3}{2}}} \tag{3.8}$$

$$1-\left(\frac{1}{2}\right)^2\frac{1}{9}+\left(\frac{1.3}{2.4}\right)^2\frac{1}{45}-... = \frac{41156608\sqrt{2\pi}}{495495\Gamma^2\left(\frac{3}{4}\right)}-\frac{16777216\sqrt{2}\Gamma^2\left(\frac{3}{4}\right)}{46475\pi^{\frac{3}{2}}} \tag{3.9}$$

$$1-\left(\frac{1}{2}\right)^2\frac{1}{10}+\left(\frac{1.3}{2.4}\right)^2\frac{1}{55}-... = \frac{153616384\sqrt{2\pi}}{935935\Gamma^2\left(\frac{3}{4}\right)}-\frac{9614393344\sqrt{2}\Gamma^2\left(\frac{3}{4}\right)}{13431275\pi^{\frac{3}{2}}} \tag{3.10}$$

(ii) SPECIAL CASES OF (2.2)

In (2.2), if we take $i = 0, 1, 2, ..., 9$, we get the following interesting results.

$$1-\left(\frac{1}{4}\right)^2+\left(\frac{1.5}{4.8}\right)^2-... = \frac{\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)}{2^{\frac{7}{4}}\pi^{\frac{3}{2}}} \tag{3.11}$$

$$1-\left(\frac{1}{4}\right)^2\frac{1}{2}+\left(\frac{1.5}{4.8}\right)^2\frac{1}{3}-... = \frac{2^{\frac{1}{4}}\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)}{3\pi^{\frac{3}{2}}}-\frac{16.2^{\frac{3}{4}}\sqrt{\pi}}{9\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)} \tag{3.12}$$

$$1-\left(\frac{1}{4}\right)^2\frac{1}{3}+\left(\frac{1.5}{4.8}\right)^2\frac{1}{6}-... = \frac{64.2^{\frac{1}{4}}\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)}{147\pi^{\frac{3}{2}}}-\frac{256.2^{\frac{3}{4}}\sqrt{\pi}}{63\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)} \tag{3.13}$$

$$1-\left(\frac{1}{4}\right)^2\frac{1}{4}+\left(\frac{1.5}{4.8}\right)^2\frac{1}{10}-... = \frac{320.2^{\frac{1}{4}}\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)}{539\pi^{\frac{3}{2}}}-\frac{19456.2^{\frac{3}{4}}\sqrt{\pi}}{2541\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)} \tag{3.14}$$

$$1-\left(\frac{1}{4}\right)^2\frac{1}{5}+\left(\frac{1.5}{4.8}\right)^2\frac{1}{15}-... = \frac{34816.2^{\frac{1}{4}}\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)}{40425\pi^{\frac{3}{2}}}-\frac{524288.2^{\frac{3}{4}}\sqrt{\pi}}{38115\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)} \tag{3.15}$$

$$1-\left(\frac{1}{4}\right)^2\frac{1}{6}+\left(\frac{1.5}{4.8}\right)^2\frac{1}{21}-... = \frac{40960.2^{\frac{1}{4}}\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)}{30723\pi^{\frac{3}{2}}}-\frac{67502080.2^{\frac{3}{4}}\sqrt{\pi}}{2751903\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)} \tag{3.16}$$

$$1-\left(\frac{1}{4}\right)^2\frac{1}{7}+\left(\frac{1.5}{4.8}\right)^2\frac{1}{28}-\ldots$$
$$=\frac{59244544.2^{\frac{1}{4}}\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)}{27087445\pi^{\frac{3}{2}}}-\frac{929038336.2^{\frac{3}{4}}\sqrt{\pi}}{21097923\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)} \quad (3.17)$$

$$1-\left(\frac{1}{4}\right)^2\frac{1}{8}+\left(\frac{1.5}{4.8}\right)^2\frac{1}{36}-\ldots$$
$$=\frac{56098816.2^{\frac{1}{4}}\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)}{14925735\pi^{\frac{3}{2}}}-\frac{58544095230.2^{\frac{3}{4}}\sqrt{\pi}}{732399327\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)} \quad (3.18)$$

$$1-\left(\frac{1}{4}\right)^2\frac{1}{9}+\left(\frac{1.5}{4.8}\right)^2\frac{1}{45}-\ldots$$
$$=\frac{671189303300.2^{\frac{1}{4}}\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)}{100405419300\pi^{\frac{3}{2}}}-\frac{3332894622000.2^{\frac{3}{4}}\sqrt{\pi}}{22704379140\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)} \quad (3.19)$$

$$1-\left(\frac{1}{4}\right)^2\frac{1}{10}+\left(\frac{1.5}{4.8}\right)^2\frac{1}{55}-\ldots$$
$$=\frac{4757078933504.2^{\frac{1}{4}}\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)}{390465519675\pi^{\frac{3}{2}}}-\frac{24053964341248.2^{\frac{3}{4}}\sqrt{\pi}}{88294807755\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)} \quad (3.20)$$

(iii) SPECIAL CASES OF (2.3)

In (2.3), if we take $i = 0, 1, 2, \ldots, 9$, we get the following interesting results.

$$1-\frac{1}{3}+\frac{1}{5}-\ldots = \frac{\pi}{4} \quad (3.21)$$
$$1-\frac{1}{5}+\frac{1.3}{5.7}-\ldots = \frac{3\pi}{4}-\frac{3}{2} \quad (3.22)$$
$$1-\frac{1}{7}+\frac{1.3}{7.9}-\ldots = \frac{15\pi}{8}-5 \quad (3.23)$$
$$1-\frac{1}{9}+\frac{1.3}{9.11}-\ldots = \frac{35\pi}{8}-\frac{77}{6} \quad (3.24)$$
$$1-\frac{1}{11}+\frac{1.3}{11.13}-\ldots = \frac{315\pi}{32}-30 \quad (3.25)$$
$$1-\frac{1}{13}+\frac{1.3}{13.15}-\ldots = \frac{693\pi}{32}-\frac{2013}{30} \quad (3.26)$$
$$1-\frac{1}{15}+\frac{1.3}{15.17}-\ldots = \frac{9009\pi}{192}-\frac{2197}{15} \quad (3.27)$$
$$1-\frac{1}{17}+\frac{1.3}{17.19}-\ldots = \frac{19305\pi}{192}-\frac{66135}{210} \quad (3.28)$$
$$1-\frac{1}{19}+\frac{1.3}{19.21}-\ldots = \frac{328185\pi}{1536}-\frac{70380}{105} \quad (3.29)$$
$$1-\frac{1}{21}+\frac{1.3}{21.23}-\ldots = \frac{230945\pi}{512}-\frac{178429}{126} \quad (3.30)$$

Clearly the result (3.2) to (3.10) are closely related to the Ramanujan's result (3.1); (3.12) to (3.20) are closely related to the Brychkov's's result (3.11) and (3.22) to (3.30) are closely related to the well known summation (3.21).

References

[1] Berndt, B. C., Ramanujan's Notebooks, Part II, *Springer, Verlage, Berlin, Heidelberg and Newyork*, (1989).

[2] Brychkov, Yury A., Hand Book of Special Functions: Derivatives, Integrals, Series and other formulas, *CRC Press*,(2008).

[3] Choi,J., Contiguous extensions of Dixon's theorem on the sum of a ${}_3F_2$, *Hindawi Publishing Corporation Journal of inequalities and applications*, Volume **2010**, Article ID – 589618, 17 pages, doI:10.1155/2010/589618.

[4] Prudnikov, A. P., Brychkov, Yury A. and Marichev, O.I., Integrals and Series, Vol.3 : More Special Functions, *Gordon and Breach, New York,* (1990).

[5] Rainville, E. D., Special Functions, *The Macmillan Company, New York,* (1960).

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