

A Note on Two Results Involving Products of Generalized Hypergeometric Functions

Junesang Choi¹

Department of Mathematics, Dongguk University
Gyeongju 38066, Republic of Korea

Arjun K. Rathie

Department of Mathematics
School of Mathematical and Physical Sciences
Central University of Kerala
Riverside Transit Campus, Padennakkad P. O. Nileshwar
Kasaragod 671 328, Kerala, India

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Abstract

Recently the authors have established the following identities:

$$e^x {}_0F_1 \left[\begin{matrix} - ; \\ \frac{1}{2} ; \end{matrix} -\frac{b^2 x^2}{4a^2} \right] = \sum_{m=0}^{\infty} \frac{x^m}{m!} \frac{(a^2 + b^2)^{m/2}}{a^m} \cos m\theta$$

and

$$e^x {}_0F_1 \left[\begin{matrix} - ; \\ \frac{3}{2} ; \end{matrix} -\frac{b^2 x^2}{4a^2} \right] = \sum_{m=1}^{\infty} \frac{x^{m-1}}{m!} \frac{(a^2 + b^2)^{m/2}}{a^m b} \sin m\theta$$

by employing two new summation formulas presented recently by Qureshi *et al.* For $b = a$, these identities reduce to the well known results due to Bailey. The objective of this note is to provide an elementary proof of these two identities.

¹Corresponding author

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1 Introduction and Preliminaries

Throughout this paper, let \mathbb{C} , \mathbb{R}_+ and \mathbb{R}_- , \mathbb{Z} and \mathbb{N} be the sets of complex numbers, positive and negative real numbers, integers and positive integers, respectively, and

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} \quad \text{and} \quad \mathbb{Z}_0^- := \mathbb{Z} \setminus \mathbb{N}.$$

It is noted that the classical summation theorems for the hypergeometric series ${}_2F_1$ such as those of Gauss and Gauss second, Kummer, and Bailey; Watson's, Dixon's, Whipple's and Saalschütz's summation theorems for the series ${}_3F_2$ and others play important roles in theory and application (see [5]).

Bailey [2] presented a large number of known, new and useful identities involving products of generalized hypergeometric series. Qureshi *et al.* [4] obtained two interesting summation formulas given in the following theorem.

Theorem 1.1. *The following formulas hold true: For $n \in \mathbb{N}_0$,*

$${}_2F_1 \left[\begin{matrix} -\frac{1}{2}n, -\frac{1}{2}n + \frac{1}{2} ; & -\frac{b^2}{a^2} \\ & \frac{1}{2} ; \end{matrix} \right] = \frac{(a^2 + b^2)^{\frac{n}{2}}}{a^n} \cos n\theta \quad (1)$$

$$\left(n \in \mathbb{N}_0; k \in \mathbb{Z}; n\theta \neq \frac{2k+1}{2}\pi \right)$$

and

$${}_2F_1 \left[\begin{matrix} -\frac{1}{2}n, -\frac{1}{2}n + \frac{1}{2} ; & -\frac{b^2}{a^2} \\ & \frac{3}{2} ; \end{matrix} \right] = \frac{(a^2 + b^2)^{\frac{n+1}{2}}}{(n+1)a^n b} \sin(n+1)\theta \quad (2)$$

$$(n \in \mathbb{N}_0; k \in \mathbb{Z}; (n+1)\theta \neq k\pi)$$

where θ is given by

$$\theta := \begin{cases} \arctan(b/a) & (a, b \in \mathbb{R}_+) \\ \pi - \arctan(b/|a|) & (a \in \mathbb{R}_-; b \in \mathbb{R}_+) \\ \arctan(b/a) - \pi & (a, b \in \mathbb{R}_-) \\ -\arctan(|b|/a) & (a \in \mathbb{R}_+; b \in \mathbb{R}_-). \end{cases} \quad (3)$$

Using (1) and (2), recently, Choi and Rathie [3] established the following two results:

$$e^x {}_0F_1 \left[\begin{matrix} -; \\ \frac{1}{2}; \end{matrix} -\frac{b^2 x^2}{4a^2} \right] = \sum_{m=0}^{\infty} \frac{x^m}{m!} \frac{(a^2 + b^2)^{m/2}}{a^m} \cos m\theta \quad (4)$$

and

$$e^x {}_0F_1 \left[\begin{matrix} -; \\ \frac{3}{2}; \end{matrix} -\frac{b^2 x^2}{4a^2} \right] = \sum_{m=1}^{\infty} \frac{x^{m-1}}{m!} \frac{(a^2 + b^2)^{m/2}}{a^m b} \sin m\theta, \quad (5)$$

where θ is the same as given in (3).

The special cases of the identities (4) and (5) when $b = a$ are seen to reduce to following results due to Bailey [2]:

$$e^x {}_0F_1 \left[\begin{matrix} -; \\ \frac{1}{2}; \end{matrix} -\frac{x^2}{4} \right] = \sum_{m=0}^{\infty} \frac{2^{m/2} x^m}{m!} \cos \frac{m\pi}{4} \quad (6)$$

and

$$e^x {}_0F_1 \left[\begin{matrix} -; \\ \frac{3}{2}; \end{matrix} -\frac{x^2}{4} \right] = \sum_{m=1}^{\infty} \frac{2^{m/2} x^{m-1}}{m!} \sin \frac{m\pi}{4}, \quad (7)$$

which were established by mainly using the following classical Kummer's summation theorem (see, *e.g.*, [5, p. 68, Eq.(2)] and [6, p. 351, Eq.(3)]):

$${}_2F_1 \left[\begin{matrix} a, b; \\ 1 + a - b; \end{matrix} -1 \right] = \frac{\Gamma(1 + a - b) \Gamma(1 + \frac{1}{2}a)}{\Gamma(1 + \frac{1}{2}a - b) \Gamma(1 + a)} \quad (8)$$

$$(\Re(b) < 1; 1 + a - b \in \mathbb{C} \setminus \mathbb{Z}_0^-).$$

It is noted that the Gauss's hypergeometric function ${}_2F_1$ and its confluent form known as the confluent hypergeometric function ${}_1F_1$ are important special functions and include most of commonly used functions as their special cases, for example, the Legendre function, the incomplete beta function, the complete elliptic functions of the first and second kinds, trigonometric functions, the Bessel functions, parabolic cylinder functions, and Coulomb wave functions. Here we recall the following two identities (see, *e.g.*, [1, p. 64] and [6, p. 73]):

$$\cos x = {}_0F_1 \left[\begin{matrix} -; \\ \frac{1}{2}; \end{matrix} -\frac{x^2}{4} \right] \quad (9)$$

and

$$\sin x = x {}_0F_1 \left[\begin{matrix} -; \\ \frac{3}{2}; \end{matrix} -\frac{x^2}{4} \right]. \quad (10)$$

Here, in this note, we aim to establish the identities (4) and (5) in a *very elementary* way.

2 Derivations of (4) and (5)

Replacing x by ax in (4) and (5) and making use of (9) and (10) with replaced x by bx , we obtain

$$e^{ax} \cos bx = \sum_{m=0}^{\infty} \frac{x^m}{m!} (a^2 + b^2)^{m/2} \cos m\theta \quad (11)$$

and

$$e^{ax} \sin bx = \sum_{m=0}^{\infty} \frac{x^m}{m!} (a^2 + b^2)^{m/2} \sin m\theta, \quad (12)$$

where θ is the same as given in (3) and which are equivalent to the identities (4) and (5), respectively. So it suffices to prove (11) and (12). Indeed, we use Euler's formula to get

$$\begin{aligned} e^{(a+ib)x} &= e^{ax} e^{ibx} \\ &= e^{ax} (\cos bx + i \sin bx) \quad (i = \sqrt{-1}). \end{aligned} \quad (13)$$

On the other hand, we expand $e^{(a+ib)x}$ as Maclurin series to obtain

$$e^{(a+ib)x} = \sum_{m=0}^{\infty} \frac{x^m}{m!} (a + ib)^m. \quad (14)$$

Let $a + ib = r e^{i\theta} = r(\cos \theta + i \sin \theta)$, where

$$r = \sqrt{a^2 + b^2} \quad \text{and} \quad \theta = \arctan \left(\frac{b}{a} \right).$$

Then (14) can be rewritten as follows:

$$e^{(a+ib)x} = \sum_{m=0}^{\infty} \frac{x^m}{m!} (a^2 + b^2)^{\frac{m}{2}} (\cos m\theta + i \sin m\theta). \quad (15)$$

We find from (13) and (15) that

$$\begin{aligned} e^{ax} (\cos bx + i \sin bx) \\ = \sum_{m=0}^{\infty} \frac{x^m}{m!} (a^2 + b^2)^{\frac{m}{2}} (\cos m\theta + i \sin m\theta). \end{aligned} \quad (16)$$

Finally, equating the real and imaginary parts of both sides of (16) is easily seen to yield the desired results (11) and (12). This completes the proof.

It is noted that the identities (11) and (12) are also recorded in [4].

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