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Article *in* Far East Journal of Mathematical Sciences · December 2016

DOI: 10.17654/MS101010119

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Far East Journal of Mathematical Sciences (FJMS)
© 2017 Pushpa Publishing House, Allahabad, India
http://www.pphmj.com
http://dx.doi.org/10.17654/MS101010119
Volume 101, Number 1, 2017, Pages 119-123 **ISSN: 0972-0871**

A NOTE ON TWO RESULTS INVOLVING PRODUCTS OF GENERALIZED HYPERGEOMETRIC FUNCTIONS

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Abstract

The objective of this note is to provide an elementary proof of two very interesting and useful results due to Bailey involving products of generalized hypergeometric functions.

1. Introduction

In the theory of hypergeometric series ${}_2F_1$ and generalized

Received: September 6, 2016; Accepted: October 4, 2016
2010 Mathematics Subject Classification: Primary 33B10, 33C20; Secondary 33C05, 33C65.
Keywords and phrases: Gamma function, Pochhammer symbol, generalized hypergeometric functions, product formulas.
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hypergeometric series ${}_pF_q$, summation formulas for ${}_pF_q$ (see, e.g., [2] and [3, Section 1.5]), in particular, the classical summation theorems for the series ${}_2F_1$, ${}_3F_2$, and ${}_4F_3$ play an important role in theory and applications (see, e.g., [2] and [3, p. 350]).

In a very useful, popular and interesting paper [1], Bailey presented a large number of known and unknown formulas involving products of generalized hypergeometric series by using the classical summation theorems for the series ${}_2F_1$ and ${}_3F_2$. Among other things, we recall the following two formulas:

$$(1-x)^{-2a}\,{}_2F_1\left[\begin{matrix} a,\ a+\frac{1}{2}; \\ \frac{1}{2}; \end{matrix}\ -\frac{x^2}{(1-x)^2}\right] = \sum_{n=0}^{\infty} 2^{\frac{n}{2}}(2a)_n \cos\left(\frac{n\pi}{4}\right)\frac{x^n}{n!} \quad (1.1)$$

and

$$(1-x)^{-2a}\,{}_2F_1\left[\begin{matrix} a,\ a+\frac{1}{2}; \\ \frac{3}{2}; \end{matrix}\ -\frac{x^2}{(1-x)^2}\right]$$

$$= \sum_{n=0}^{\infty} 2^{\frac{n+1}{2}}(2a)_n \sin\left(\frac{(n+1)\pi}{4}\right)\frac{x^n}{(n+1)!}, \quad (1.2)$$

where $(\lambda)_n$ is the Pochhammer symbol defined $(\text{for } \lambda \in \mathbb{C})$ by (see [3, p. 2 and p. 5]):

$$(\lambda)_n := \begin{cases} 1 & (n = 0) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (n \in \mathbb{N}) \end{cases}$$

$$= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C}\setminus\mathbb{Z}_0^-) \quad (1.3)$$

and $\Gamma(\lambda)$ is the familiar Gamma function. Here and in what follows, $\mathbb{C}$, $\mathbb{N}$ and $\mathbb{Z}_0^-$ denote the sets of complex numbers, positive and non-positive integers, respectively.

Bailey [1] obtained the formulas (1.1) and (1.2) by employing the following classical Kummer's summation theorem (see, e.g., [2, p. 68]):

$$ {}_2F_1\left[\begin{matrix} a, b; \\ 1 + a - b; \end{matrix} \; -1\right] = \frac{\Gamma\left(1 + \frac{1}{2}a\right)\Gamma(1 + a - b)}{\Gamma(1 + a)\Gamma\left(1 + \frac{1}{2}a - b\right)} \tag{1.4} $$

$$ (\Re(b) < 1;\ 1 + a - b \in \mathbb{C}\backslash\mathbb{Z}_0^-). $$

The objective of this note is to show how the formulas (1.1) and (1.2) can be derived in an elementary way.

2. Derivations of (1.1) and (1.2)

We first assume that the involved parameter *a* and the variable *x* in (1.1) and (1.2) are real numbers. Let $\mathcal{L}$ be the right-hand side of (1.1). Then we find that

$$ \mathcal{L} = \Re\left(\sum_{n=0}^{\infty} (2a)_n \frac{\left(\sqrt{2}xe^{\frac{i\pi}{4}}\right)^n}{n!}\right) \qquad (i = \sqrt{-1}) $$

$$ = \Re\left(\sum_{n=0}^{\infty} (2a)_n \frac{(x(1 + i))^n}{n!}\right) $$

which, in terms of generalized hypergeometric function, can be written as

$$ \mathcal{L} = \Re\left({}_1F_0\left[\begin{matrix} 2a; \\ -; \end{matrix} \; x(1 + i)\right]\right). \tag{2.1} $$

Applying the following well known formula:

$$ {}_1F_0\left[\begin{matrix} a; \\ -; \end{matrix}\; z\right] = (1-z)^{-a} \tag{2.2}$$

to (2.1), we obtain

$$\begin{aligned} L &= \Re([1 - x(1+i)]^{-2a}) \\ &= (1-x)^{-2a}\,\Re\left(\left[1 - \frac{ix}{1-x}\right]^{-2a}\right) \\ &= (1-x)^{-2a}\,\Re\left({}_1F_0\left[\begin{matrix} 2a; \\ -; \end{matrix}\; \frac{ix}{1-x}\right]\right), \end{aligned} \tag{2.3}$$

where (2.2), again, is used for the last equality.

We see that

$$\begin{aligned} {}_1F_0\left[\begin{matrix} 2a; \\ -; \end{matrix}\; \frac{ix}{1-x}\right] &= \sum_{n=0}^{\infty} \frac{(2a)_n}{n!}\left(\frac{ix}{1-x}\right)^n \\ &= \sum_{n=0}^{\infty} \frac{(2a)_{2n}}{(2n)!}(-1)^n\left(\frac{x}{1-x}\right)^{2n} \\ &\quad + i\sum_{n=0}^{\infty} \frac{(2a)_{2n+1}}{(2n+1)!}(-1)^n\left(\frac{x}{1-x}\right)^{2n+1}. \end{aligned} \tag{2.4}$$

We find from (2.3) and (2.4) that

$$\mathcal{L} = (1-x)^{-2a}\sum_{n=0}^{\infty} \frac{(2a)_{2n}}{(2n)!}(-1)^n\left(\frac{x}{1-x}\right)^{2n},$$

which, upon using the following elementary identities:

$$(2a)_{2n} = 2^{2n}(a)_n\left(a+\frac{1}{2}\right)_n$$

and

$$(2n)! = 2^{2n}\left(\frac{1}{2}\right)_n n!,$$

becomes

$$\mathcal{L} = (1-x)^{-2a}\sum_{n=0}^{\infty}\frac{(a)_n\left(a+\frac{1}{2}\right)_n(-1)^n}{\left(\frac{1}{2}\right)_n n!}\left(\frac{x}{1-x}\right)^{2n}. \tag{2.5}$$

The expression in (2.5) is just the left-hand side of (1.1).

A similar argument will establish (1.2) by taking imaginary part. Its detailed account is omitted.

Next, the formulas (1.1) and (1.2) when the involved parameter a and the variable x are complex numbers are easily seen to hold true by the principle of analytic continuation.

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