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Some Results for Poisson and Beta distributions

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Abstract

Explicit expressions of probability functions and probability generating functions for mixed Poisson distributed discrete random variables are given corresponding to the following structure density functions: generalized gamma, generalized shifted gamma and generalized shifted beta. A discrete symmetric distribution corresponding to a stochastic process is approximated by a beta distribution in a more accurate manner. A generalized Beta-Poisson distribution is obtained. The results are useful in biological and economical problems. Special cases are also mentioned. Graphs are drawn for probability functions showing the modality for different values of the parameters. Transition intensities can be easily obtained for the various cases discussed in this paper. Finally, by utilizing the fact that probabilities sum to 1, we obtain some new results for generalized hypergeometric functions.

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1 Introduction

A discrete random variable N is defined to be mixed Poisson distributed with structure density function $u(l)$ if (see, [6], p. 13)

$$p_k(t) = (k!)^{-1} \int_0^\infty (lt)^k e^{-lt} u(l) dl, \quad k = 0, 1, 2, \ldots, \quad t > 0. \tag{1}$$

The probability generation function $G_N(s)$ of N is given by

$$G_N(s) := E(s^N) = \sum_{k=0}^{\infty} s^k p_k(t) = \int_0^\infty e^{-t(1-s)l} u(l) dl, \quad s \leq 1. \tag{2}$$

Examples of (1) with structure distributions gamma (Bühlmann [4]), including exponential and Erlang truncated or shifted gamma (Delaporte [5], Ruohonen [16], Willmot and Sundt [21], Schröter [18], Willmot [22]), generalized inverse Gaussian (Sichel [19], Willmot [23]), beta (Gurland [7], McNolty [14], Beall and Rescia [1], Bhattacharya and Holla [2]), truncated normal (Berljand et al. [3], Kupper [[11], [12]], Patil [15]) are available in the literature. Mixed Poisson distributions are used in biological and other applications (see, [6]).

Johnson and Kotz ([8], p. 52) comment that little work has been done on "Weibullized beta distributions". One of the object of this paper is to answer this question satisfactorily, in detail, along with providing other results on the topic. Software R is employed to draw graphs for $p_k(t)$ in various cases. The graphs show the change of modality for different values of the parameters.

The results for mixed Zero Inflated Poisson (ZIP) distribution (Johnson and Kotz [8], p. 205, eq. (70)) can be easily obtained by utilizing the following definition

$$p_k(t) = \begin{cases} w + (1-w) p_0(t), & k = 0; \\ (1-w) p_k(t), & k = 1, 2, \ldots, \end{cases} \tag{3}$$

where $0 \leq w < 1$ and $p_k(t)$ is defined in (1).

The transition intensities $k_n(t)$ ([6], p. 62) are given by

$$k_n(t) = \frac{\int_0^\infty l^{n+1} e^{-lt} u(l) dl}{\int_0^\infty l^n e^{-lt} u(l) dl},$$

which on using (1) yields

$$k_n(t) = \frac{n+1}{t} \frac{p_{n+1}(t)}{p_n(t)}. \tag{4}$$

Thus $k_n(t)$, for the various cases discussed in this paper, is easily obtainable from the expression for $p_k(t)$.

The following results (see Mathai et al. [13]; Springer, [20]) will be used later on

1. Result 1:

$$\int_0^\infty z^{\alpha-1} e^{\beta z - \gamma z^\delta} dz = \beta^{-\alpha} H_{1,1}^{1,1}\left[\frac{\gamma}{\beta^\delta}\,\middle|\, \begin{matrix}(1-\alpha,\delta)\\(0,1)\end{matrix}\right], \tag{5}$$

for $\alpha,\ \beta,\ \gamma,\ \delta > 0$.

2. Result 2:

$$\Gamma(-a)(1+z)^a = G_{1,1}^{1,1}\left[z\,\middle|\, \begin{matrix}1+a\\0\end{matrix}\right]. \tag{6}$$

3. Result 3:

$$\sum_{n=0}^{\infty} \frac{\Gamma(a_1+nA_1)}{\Gamma(b_1+nB_1)}\frac{z^n}{n!} = H_{1,2}^{1,1}\left[-z\,\middle|\, \begin{matrix}(1-a_1,A_1)\\(0,1),(1-b_1,B_1)\end{matrix}\right]. \tag{7}$$

The paper is divided as follows: Section 2 deals with generalized gamma-Poisson distribution, while Section 3 is devoted to generalized (shifted) beta-Poisson distribution. The equilibrium symmetric distribution $\mu(k)$ studied by Kirman [10] and its approximation by a beta distribution are treated in Section 4. In Section 5, generalized beta-Poisson distribution is studied. Last, Section 6, mentions a few results involving hypergeometric functions.

2 Generalized Gamma-Poisson Distribuion

We start defining the generalized gamma distribution with density function $u(l)$ given by

$$u(l) = \left[\frac{\delta\beta^{\gamma/\delta}}{\Gamma(\gamma/\delta)}\right] l^{\gamma-1} e^{-\beta l^\delta},\ \ l,\beta,\gamma,\delta > 0. \tag{8}$$

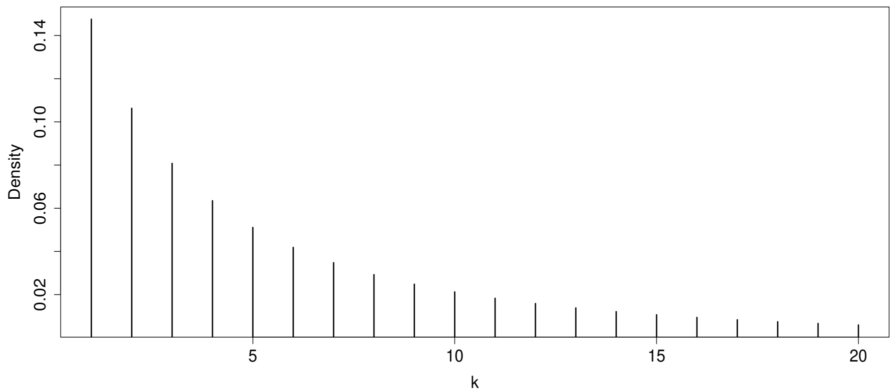
Then, from (1), the generalized gamma-Poisson distribution has the probability function given by

$$\begin{aligned} p_k(t) &= \frac{\delta\beta^{\gamma/\delta} t^k}{\Gamma(\gamma/\delta)k!}\int_0^\infty l^{k+\gamma-1} e^{-tl} e^{-\beta l^\delta} dl & (9)\\ &= \frac{\delta\beta^{\gamma/\delta} t^{-\gamma}}{\Gamma(\gamma/\delta)k!} H_{1,1}^{1,1}\left[\frac{\beta}{t^\delta}\,\middle|\, \begin{matrix}(1-k-\gamma,\delta)\\(0,1)\end{matrix}\right], & (10)\end{aligned}$$

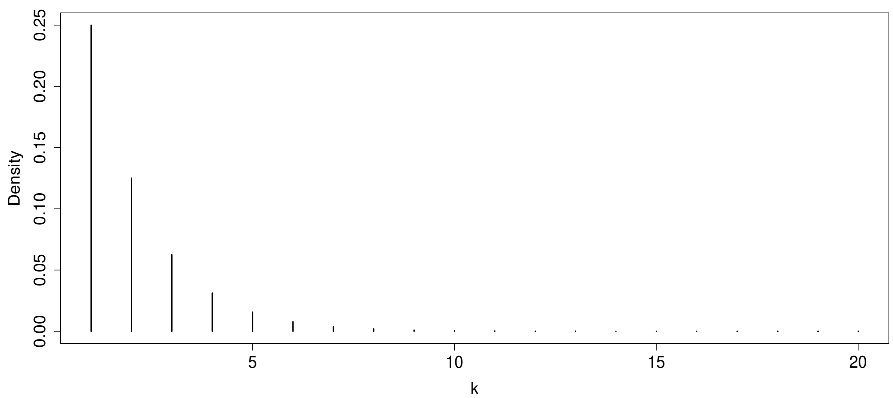
by utilizing (5). For $\delta = 1$, (10) yields, with the help of (6), the following result

$$p_k(t) = \frac{\beta^\gamma t^k \Gamma(k+r)}{k!\Gamma(\gamma)(t+\beta)^{k+\gamma}}, \tag{11}$$

which agrees with Grandell ([6], p. 17). Note that there is no need for γ to be an integer as commented by Grandell.



(a) $\gamma = \beta = t = 1$ and $\delta = 0.5$.



(b) $\gamma = \beta = t = \delta = 1$

Figure 1: Plots of (10) for some values of the parameters

The probability generating function, $G_N(s)$, for the generalized gamma distribution (8) can be similarly obtained from (2) as

$$G_N(s) = \int_0^\infty e^{-t(1-s)l} u(l) dl = \frac{\delta \beta^{\gamma/\delta}}{\Gamma(\gamma/\delta)} \int_0^\infty l^{\gamma-1} e^{-t(1-s)l - \beta l^\delta} dl \qquad (12)$$

$$= \frac{\delta \beta^{\gamma/\delta}}{\Gamma(\gamma/\delta)} (t(1-s))^{-\gamma} H_{1,1}^{1,1} \left[\frac{\beta}{(t(1-s))^\delta} \,\middle|\, \begin{matrix} (1-\gamma, \delta) \\ (0,1) \end{matrix} \right]. \qquad (13)$$

For $\delta = 1$, (13) reduces to the known probability generating function given in Grandell ([6], p. 33). Plots of (10) for some values of the parameters, showing modes, are given in Figure 1.

3 Generalized (Shifted) Beta-Poisson Distribution

Let us define the generalized (Shifted) beta distribution with density function $u(l)$ given by

$$u(l) = \frac{(l-a)^{\alpha-1}(b-l)^{\beta-1}}{B(\alpha,\beta)(b-a)^{\alpha+\beta-1}}, \tag{14}$$

with $a \leq l \leq b$ and $\alpha, \beta > 0$. Then, from (1), the generalized (Shifted) beta-Poisson distribution has the probability function given by

$$p_k(t) \;=\; \int_a^b \frac{(lt)^k e^{-lt}}{k!} \frac{(l-a)^{\alpha-1}(b-l)^{\beta-1}}{B(\alpha,\beta)(b-a)^{\alpha+\beta-1}} dl,$$

by substituting $y = (l-a)/(b-a)$ we have

$$\begin{aligned} p_k(t) &= \frac{t^k e^{-at}}{k! B(\alpha,\beta)} \int_0^1 [a+(b-a)y]^k e^{-t(b-a)y} y^{\alpha-1}(1-y)^{\beta-1} dy \\ &= \frac{t^k e^{-at}}{k! B(\alpha,\beta)} \sum_{n=0}^{k} \binom{k}{n} a^{k-n}(b-a)^n \int_0^1 y^{\alpha+n-1}(1-y)^{\beta-1} e^{-t(b-a)y} dy \\ &= \frac{t^k e^{-at}}{k! B(\alpha,\beta)} \sum_{n=0}^{k} \binom{k}{n} a^{k-n}(b-a)^n B(\alpha+n,\beta) \times \\ &\times \;\; {}_1F_1(\alpha+n; \alpha+\beta+n; -t(b-a)), \end{aligned} \tag{15}$$

by utilizing (Luke [17], p. 115, eq. (1)), where ${}_1F_1$ is the confluent hypergeometric function. For $a = 0$, (15) yields a known result ([6], p. 45). The corresponding probability generating function is given by

$$G_N(s) = e^{-at(1-s)} {}_1F_1(\alpha; \alpha+\beta; -t(1-s)(b-a)). \tag{16}$$

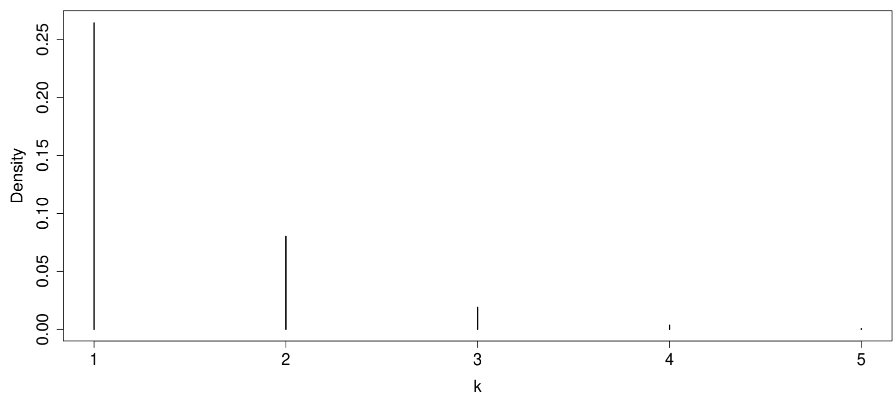
Plots of (15) for some values of the parameters, showing modes, are given in Figure 2.

4 Beta distribution

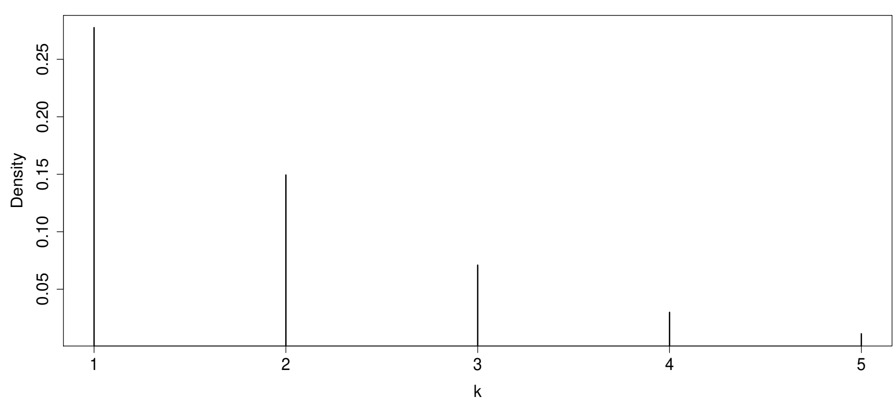
Kirman [10] has obtained the equilibrium symmetric distribution $\mu(k)$, $k = 0, 1, 2, \ldots, N$, of the Markov chain defined for the process

$$k \to \begin{cases} k+1 \text{ with probability } p_1 = p(k,k+1) = \left(1-\frac{k}{N}\right)\left(\epsilon + (1-\delta)\frac{k}{N-1}\right) \\ k-1 \text{ with probability } p_2 = p(k,k-1) = \frac{k}{N}\left(\epsilon + (1-\delta)\frac{N-k}{N-1}\right) \\ k \text{ with probability } 1-p_1-p_2, \end{cases} \tag{17}$$

in the following form



(a) $a = 0$ and $b = \alpha = \beta = t = 1$.



(b) $a = 0$, $b = 1$, $\alpha = 1$, $\beta = 2$ and $t = 3$.

Figure 2: Plots of (15) for some values of the parameters

$$\frac{\mu(k+1)}{\mu(k)} = \frac{\left(1-\frac{k}{N}\right)\left(\epsilon+(1-\delta)\frac{k}{N-1}\right)}{\left(\frac{k+1}{N}\right)\left(\epsilon+(1-\delta)\left(1-\frac{k}{N-1}\right)\right)} \tag{18}$$

for $k = 0, 1, \ldots, N-2$, $N > 2$, $0 \leq \epsilon \leq \delta \leq 1$. Also

$$\mu(k) = \sum_{i=0}^{N} \mu(i)p(i,k), \quad \sum_{i=0}^{N} \mu(i) = 1, \tag{19}$$

and the reversible relation

$$\mu(k)p(k,i) = \mu(i)p(i,k). \tag{20}$$

By utilizing (18) in the following expression

$$\frac{\mu(k)}{\mu(0)} = \frac{\mu(1)}{\mu(0)}\frac{\mu(2)}{\mu(1)}\cdots\frac{\mu(k)}{\mu(k-1)},$$

we obtain

$$\frac{\mu(k)}{\mu(0)} = \prod_{i=0}^{k-1} \frac{\left(1-\frac{i}{N}\right)\left(\epsilon+(1-\delta)\frac{i}{N-1}\right)}{\left(\frac{i+1}{N}\right)\left(\epsilon+(1-\delta)\left(1-\frac{i}{N-1}\right)\right)}, \tag{21}$$

where an empty product is interpreted as 1 and $\epsilon > 0$. For $\delta = 1$, (21) reduces to

$$\mu_k = \binom{N}{k} 2^{-N}, \quad k = 0, 1, \ldots, N, \tag{22}$$

which is Ehrenfest urn model.

From (20), we obtain easily the following recurrence relation

$$\mu(j+1)p(j+1,j) = \mu(j)[1-p(j,j)] - \mu(j-1)p(j-1,j), \quad j = 0, 1, \ldots. \tag{23}$$

Now, we approximate $\mu(k)$ by a beta distribution. Let $\alpha = \epsilon N$ and redefining $\mu(k)$ as $f(k/N)$, we can approximate $f(k/N)$ by $f(x)$ as $N \to \infty$, where $x = k/N \in [0,1]$. Thus, by (18), we have

$$\begin{aligned} \frac{f'(x)}{f(x)} &= \lim_{N\to\infty} N\frac{\mu(k+1)-\mu(k)}{\mu(k)} = \frac{(\alpha-1)(1-2x)+\delta(1-x)-\epsilon}{(1-\delta)x(1-x)} \\ &= \frac{\alpha+\delta-\epsilon-1}{1-\delta}\frac{1}{x} - \frac{\alpha+\epsilon-1}{1-\delta}\frac{1}{1-x}, \end{aligned}$$

giving

$$f(x) = \frac{1}{B\left(\frac{\alpha-\epsilon}{1-\delta}, \frac{\alpha-\delta+\epsilon}{1-\delta}\right)} x^{\frac{\alpha-\epsilon}{1-\delta}-1}(1-x)^{\frac{\alpha-\delta+\epsilon}{1-\delta}-1}, \tag{24}$$

for $0 < x < 1$, $\alpha > 0$, $\delta - \alpha < \epsilon < \alpha$ and $0 \leq \epsilon \leq \delta \leq 1$. For (24) to be symmetric, we require $\delta = 2\epsilon$, giving

$$X \sim B\left(\frac{\alpha-\epsilon}{1-2\epsilon}, \frac{\alpha-\epsilon}{1-2\epsilon}\right), \quad 0 \leq \epsilon \leq \alpha. \tag{25}$$

For $\epsilon = 0$, (25) yields Kirman's result ([10], p. 146).

There are three special cases of (25) for $\frac{\alpha-\epsilon}{1-2\epsilon} < 1$, $\frac{\alpha-\epsilon}{1-2\epsilon} > 1$ and $\frac{\alpha-\epsilon}{1-2\epsilon} = 1$, i.e, $\alpha < 1-\epsilon$, $\alpha > 1-\epsilon$ and $\alpha = 1-\epsilon$. For the first one (25) is U-shaped beta. For the second case, (25) is uniform and for the last case (25) unimodal-shaped beta. U-shaped beta is useful in economica applications [10].

5 Generalized Beta-Poisson Distribution

This section deals with the generalized beta-Poisson distribution which generalizes the results given earlier by Bhattacharya and Holla [2] and Grandell [6].

Theorem 5.1 *Let $Y = aX^b$, $a > 0$, $b > 0$ and $X \sim Beta(\alpha, \beta)$. Then density function $u(y)$ of Y, which may be called as generalized beta distribution is given by*

$$u(y) = \frac{y^{\frac{\alpha}{b}-1}\left[1-(y/a)^{1/b}\right]^{\beta-1}}{a^{\frac{\alpha}{b}} b B(\alpha,\beta)}, \quad 0 \leq y \leq a, \tag{26}$$

and

$$p_k(t) = \frac{(at)^k}{k! B(\alpha,\beta)} \sum_{r=0}^{\infty} \frac{(-at)^r}{r!} B(b(r+k)+\alpha, \beta), \tag{27}$$

for $t \geq 0, k = 0, 1, 2, \ldots$. Alternatively, we can write $p_k(t)$ as

$$p_k(t) = \frac{(at)^k \Gamma(\beta)}{k! B(\alpha,\beta)} H_{1,2}^{1,1}\left[at \middle| \begin{matrix} (1-bk-\alpha, b) \\ (0,1), (1-bk-\alpha-\beta, b) \end{matrix}\right] \tag{28}$$

We may call $p_k(t)$ as a generalized beta-Poisson density.

Proof:

Substituting $u(y)$ in (1), utilizing $e^{-tu} = \sum_{r=0}^{\infty} \frac{(-tu)^r}{r!}$ and integrating with the help of beta function of first kind, we arrive at (27). The equation (28) is obtained on utilizing (27) and (7).

5.1 Special Cases

1. For $\beta = 1$, (27) gives
$$p_k(t) = \frac{\alpha(at)^k}{k!(kb+\alpha)} {}_1F_1\left(k+\frac{\alpha}{b}; k+\frac{\alpha}{b}+1; -at\right), \tag{29}$$
where ${}_1F_1$ is the confluent hypergeometric function. For $b = \alpha$, the above result reduces to the following result given by Bhattacharya and Holla [2]
$$p_k(t) = \frac{\gamma(k+1, at)}{k!at} \tag{30}$$
where $\gamma(.,.)$ is the incomplete gamma function.

2. For $b = 1$, (27) yields
$$p_k(t) = \frac{(at)^k B(k+\alpha, \beta)}{k!B(\alpha, \beta)} {}_1F_1\left(k+\alpha; k+\alpha; \beta; -at\right). \tag{31}$$
This is given in Grandell ([6], p. 45) and is useful in biological applications. For $\beta = 1$, the above results gives
$$p_k(t) = \frac{\alpha\gamma(k+\alpha, at)}{k!(at)^\alpha} \tag{32}$$

3. Taking $b = m$ (a positive integer) in (27) and using multiplication formula for gamma function Luke [17], we can express $p_k(t)$ in terms of the generalized hypergeometric function ${}_mF_m$ as follows
$$p_k(t) = \frac{(at)^k\Gamma(\beta)}{k!m^\beta B(\alpha,\beta)} \prod_{j=0}^{m-1} \frac{\Gamma\left(k+\frac{\alpha+j}{m}\right)}{\left(k+\frac{\alpha+\beta+j}{m}\right)} \times$$
$${}_mF_m\left(k+\frac{\alpha}{m}, \ldots, k+\frac{\alpha+m-1}{m}; k+\frac{\alpha+\beta}{m}, \ldots, k+\frac{\alpha+\beta+m-1}{m}; -at\right) \tag{33}$$

4. For $\alpha = \beta$ (symmetrical beta), (27) gives
$$p_k(t) = \frac{(at)^k}{k!B(\alpha,\alpha)} \sum_{r=0}^{\infty} \frac{(-at)^r}{r!} B(b(r+k)+\alpha, \alpha). \tag{34}$$
When $\alpha = n$, a positive integer, $p_k(t)$ can be expressed as a linear combination of incomplete gamma functions. For this, we write
$$\begin{aligned} B(c+n,n) &= \frac{\Gamma(c+n)\Gamma(n)}{\Gamma(c+2n)} \\ &= \frac{\Gamma(n)}{(c+2n-1)(c+2n-2)...(c+n)} \\ &= \Gamma(n)\sum_{i=1}^{n} \frac{a_i}{(c+n-1+i)}, \end{aligned} \tag{35}$$

where $c = b(r+k)$ and a_i is obtained by utilizing the partial fraction procedure. Thus

$$\begin{aligned} p_k(t) &= \frac{\Gamma(n)(at)^k}{k!B(n,n)} \sum_{i=1}^{n} a_i \sum_{r=0}^{\infty} \frac{(-at)^r}{r!(b(r+k)+n-1+i)} \\ &= \frac{\Gamma(n)(at)^k}{k!bB(n,n)} \sum_{i=1}^{n} a_i \sum_{r=0}^{\infty} \frac{(-at)^r}{r!\left(r+k+\frac{n-1+i}{b}\right)} \\ &= \frac{\Gamma(n)(at)^k}{k!bB(n,n)} \sum_{i=1}^{n} \frac{1}{k+\frac{n-1+i}{b}} a_{i\,1}F_1\left(k+\frac{n-1+i}{b}; k+\frac{n-1+i}{b}+1; -at\right) \\ &= \frac{\Gamma(n)}{k!bB(n,n)} \sum_{i=1}^{n} (at)^{-\frac{n-1+i}{b}} a_i \gamma\left(k+\frac{n-1+i}{b}, at\right) \end{aligned} \tag{36}$$

where

$$a_i = \frac{(-1)^{i-1}}{(i-1)!(n-i)!}. \tag{37}$$

5. When we take $\beta = n$, where n is a positive integer, in (27), we get

$$p_k(t) = \frac{\Gamma(n)}{k!bB(\alpha,n)} \sum_{i=1}^{n} a_i (at)^{-\frac{\alpha-1+i}{b}} \gamma\left(k+\frac{\alpha-1+i}{b}; at\right), \tag{38}$$

where a_i is given in (37). For $\alpha = n$, (38) reduces to (36).

6 Some new results for Generalized Hypergeometric Functions

A few results for generalized hypergeometric functions are easily obtainable from the results of the last section. Since $\sum_{k=0}^{\infty} p_k(t) = 1$ we have the following result from (28)

$$H_{1,1}^{1,2}\left[at \,\middle|\, \begin{matrix} (1-bk-\alpha, b) \\ (0,1), (1-bk-\alpha-\beta, b) \end{matrix}\right] = \frac{k!\Gamma(\alpha)(at)^{-k}}{\Gamma(\alpha+\beta)}. \tag{39}$$

In a similar manner, we have the following results respectively from (29) and (33)

1. Result 1:

$$\sum_{k=0}^{\infty} \frac{(at)^k}{k!(kb+\alpha)} {}_1F_1\left(k+\frac{\alpha}{b}; k+\frac{\alpha}{b}+1; -at\right) = \frac{1}{\alpha} \tag{40}$$

2. Result 2:

$$\frac{m^\beta \Gamma(\alpha)}{\Gamma(\alpha+\beta)} = \sum_{k=0}^{\infty} \frac{(at)^k}{k!} \prod_{j=0}^{m-1} \frac{\Gamma(k+\frac{\alpha+j}{m})}{\Gamma(k+\frac{\alpha+\beta+j}{m})} \times$$
$$\times {}_mF_m\left(k+\frac{\alpha}{m}; k+\frac{\alpha+1}{m}, \ldots, k+\frac{\alpha+m-1}{m}; k+\frac{\alpha+\beta}{m}, k+\frac{\alpha+\beta+1}{m}, \ldots, k+\frac{\alpha+\beta+m-1}{m}; -at\right) \quad (41)$$

where m is a positive integer.

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References

[1] G. Beall and R.R. Rescia. A generalization of Neyman's contagious distributions. *Biometrics*, 9: 354-386, 1953.

[2] S.K. Bhattacharya and M.S. Holla. On a discrete distribution with special reference to the theory of accident proneness. *Journal of the American Statistical Association*, 60: 1060-1066, 1965.

[3] O.S. Berljand, I.M. Nazarov, and A.J. Pressman. i^n $erfc$-distribution or mixed Gaussian-Poisson distribution. (Russian) *Dokl. Akad. Nauk SSSR*, 147: 1005-1007, 1962.

[4] H. Bühlmann. *Mathematical Methods in Risk Theory*. Springer-Verlag, Berlin, 1970.

[5] P. Delaporte. Un problem de trafication de I'assurance accidents d'automobile examin par la statistique mathématique. *Trans. of the XVIth International Congress of Actuaries II*, Brussels, 121-135, 1960.

[6] J. Grandell. *Mixed Poisson Processes*. Chapman & Hall, London, 1997.

[7] J. Gurland. A generalized class of contagious distributions. *Biometrics*, 14: 229-249, 1958.

[8] N.L. Johnson and S. Kotz. *Discrete Distributions*. Wiley, New York, 1969.

[9] N.L. Johnson and S. Kotz. *Continuous Univariate Distributions*. Vol. 2, Wiley, New York, 1970.

[10] A. Kirman. Ants, Rationality, and Recruitment. *The Quarterly Journal of Economics*, 108: 137-156, 1993.

[11] J. Kupper. Wahrscheinlichkeitstheoretische Modelle in der Schadenversicherung, Teil I: Die Schadenzahl. *Blätter der deutschen Gesellschaft für Versicherungsmathematik*, 5: 451-503, 1962.

[12] J. Kupper. Wahrscheinlichkeitstheoretische Modelle in der Schadenversicherung, Teil II: Schadenhöhe und Totalschaden. *Blätter der deutschen Gesellschaft für Versicherungsmathematik*, 6: 95-130, 1962.

[13] Mathai, A.M.; Saxena, R.K.; Haubold, H.J. *The H-Function: Theory and Applications*; Springer, New York, 2010.

[14] F. McNolty. Reliability density functions when the failure rate is randomly distributed. *Sankhya A*, 26: 287-292, 1964.

[15] G.P. Patil. On certain compound Poisson and compound binomial distributions. *Sankhya A*, 26: 293-294, 1964.

[16] M. Ruohonen. On a model for the claim number process. *Astin Bulletin*, 18: 57-68, 1988.

[17] Luke, Y.L. *The Special Functions and Their Approximations*, Vol 1, Academic Press, New York, 1969.

[18] K.J. Schröter. On a family of counting distributions and recursions for related compound Poisson. *Scandinavian Actuarial Journal*, 1990: 161-175, 1990.

[19] H.S. Sichel. On a family of discrete distributions particular suited to represents long tailed frequency data. *Proceedings of the Third Symposium on Mathematical Statistics*. Ed. by N.F. Laubscher, CSIR, Pretorial, 51-97.

[20] Springer, M.D. *Algebra of Random Variables*; John Wiley, New York, 1979.

[21] G. Willmot. The Poisson-Inverse Gaussian distribution as an alternative to the Negative Binomial. *Scandinavian Actuarial Journal*, 1987: 113-127, 1987.

[22] G. Willmot. On recursive evaluation of mixed Poisson probabilities and related quantities. *Scandinavian Actuarial Journal*, 1993: 114-133, 1993.

[23] G. Willmot and B. Sundt. On evaluation of the Delaporte distribution and related distributions. *Scandinavian Actuarial Journal*, 1989: 101-113, 1989.

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