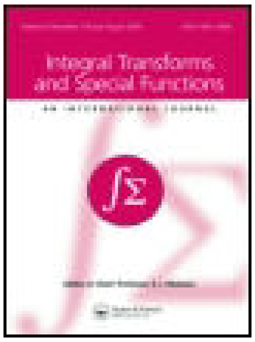

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Research Article

On new reduction formulas for the Humbert functions Ψ_2, Φ_2 and Φ_3

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ABSTRACT
New reduction formulas for the Humbert functions (the confluent Appell functions) Ψ_2, Φ_2 and Φ_3 are given.

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1. Introduction

The Humbert functions Ψ_2, Φ_2 and Φ_3 (confluent forms of the Appell hypergeometric functions of two variables) are defined by the series (see, e.g. [1–5])

$$\Psi_2(a;\, c,\, c';\, w,\, z) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{(a)_{k+m}}{(c)_k (c')_m} \frac{w^k z^m}{k!\,m!}, \tag{1.1}$$

$$\Phi_2(b,\, b';\, c;\, w,\, z) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{(b)_k (b')_m}{(c)_{k+m}} \frac{w^k z^m}{k!\,m!} \tag{1.2}$$

and

$$\Phi_3(b;\, c;\, w,\, z) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{(b)_k}{(c)_{k+m}} \frac{w^k z^m}{k!\,m!} \tag{1.3}$$

which converge absolutely at any $w, z \in \mathbb{C}$.

Some relations between the Humbert functions and hypergeometric functions are available in the literature [2,3,6–9]. These are

$$\Psi_2(a;\, a,\, a;\, w,\, z) = e^{w+z}\, {}_0F_1\left(a;\, wz\right), \tag{1.4}$$

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$$\Psi_2(a;\, c,\, c;\, z,\, -z) = {}_2F_3\left(\begin{matrix} \frac{a}{2}, \frac{a+1}{2} \\ c, \frac{c}{2}, \frac{c+1}{2} \end{matrix}; -z^2\right), \tag{1.5}$$

$$\Psi_2(a;\, c,\, c';\, z,\, z) = {}_3F_3\left(\begin{matrix} a, \frac{c+c'-1}{2}, \frac{c+c'}{2} \\ c, c', c+c'-1 \end{matrix}; 4z\right), \tag{1.6}$$

$$\Phi_2(b,\, b';\, b+b';\, w,\, z) = e^z{}_1F_1\left(\begin{matrix} b \\ b+b' \end{matrix}; w-z\right), \tag{1.7}$$

$$\Phi_2(b,\, b';\, c;\, z,\, z) = {}_1F_1\left(\begin{matrix} b+b' \\ c \end{matrix}; z\right), \tag{1.8}$$

$$\Phi_2(a,\, a;\, c;\, z,\, -z) = {}_1F_2\left(\begin{matrix} a \\ \frac{c}{2}, \frac{c+1}{2} \end{matrix}; \frac{z^2}{4}\right). \tag{1.9}$$

Reduction formulas for

$$\Phi_2\left(b, m; 2b+m-n; z, \frac{z}{2}\right), \quad \Phi_2(b+m, b'+n; b+b'-p; w, z),$$

$$\Phi_2(2n, 1; 4n+1; w, z),$$

$$\Phi_2(2n, 1; 4n+3; w, z), \quad \Phi_3\left(m+1; \frac{3}{2}-n; w, z\right), \quad \Phi_3\left(m+1; n+\frac{3}{2}; w, z\right),$$

where m, n and p are arbitrary non-negative integers, can be found in [10].

The following expansions which are useful for derivation of new reduction formulas were obtained [11,12]:

(1) For $c, c' \neq 0, -1, -2, \ldots$,

$$\Psi_2(a;\, c, c';\, w,\, z) = \sum_{m=0}^{\infty} \frac{(a)_m}{(c)_m}\, {}_2F_1\left(\begin{matrix} -m,\, -m-c+1 \\ c' \end{matrix}; \frac{z}{w}\right) \frac{w^m}{m!}; \tag{1.10}$$

(2) For $c \neq 0, -1, -2, \ldots$,

$$\Phi_2(b,\, b';\, c;\, w,\, z) = \sum_{m=0}^{\infty} \frac{(b)_m}{(c)_m}\, {}_2F_1\left(\begin{matrix} -m,\, b' \\ 1-b-m \end{matrix}; \frac{z}{w}\right) \frac{w^m}{m!}; \tag{1.11}$$

(3)

$$\Phi_3(b;\, c;\, w,\, z) = e^{w+z/w} \sum_{m=0}^{\infty} \frac{(-\frac{z}{w})^m}{m!}\, {}_2F_1\left(\begin{matrix} -m, b-c-m+1 \\ c \end{matrix}; \frac{w^2}{z}\right); \tag{1.12}$$

some special cases are considered. It is not out of place to mention here that the expansion for Φ_3,

$$\Phi_3(b;\, c;\, w,\, z) = \sum_{m=0}^{\infty} \frac{(b)_m}{(c)_m}\, {}_1F_1\left(\begin{matrix} -m \\ 1-b-m \end{matrix}; \frac{z}{w}\right) \frac{w^m}{m!}, \tag{1.13}$$

follows from (1.12) due to the confluence formulas [11,12].By appropriate application of the Gauss formula [13]

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \mathrm{Re}(c-a-b) > 0, \tag{1.14} $$

or the Kummer formula [13]

$$ {}_2F_1\left(\begin{matrix} a, b \\ a-b+1 \end{matrix}; -1\right) = \frac{\Gamma(\frac{a}{2}+1)\Gamma(a-b+1)}{\Gamma(a+1)\Gamma(\frac{a}{2}-b+1)}, \mathrm{Re}\, b < 1, \tag{1.15} $$

we can easily establish the results (1.5), (1.6) and (1.9).

The main purpose of this paper is to obtain several general reduction formulas for Humberts functions Ψ_2, Φ_2 and Φ_3.

The reduction formulas will be derived with the help of following generalizations of the Kummer formula (1.15) written in a slightly modified form [14] where $n = 0, 1, 2, \ldots$:

$$ {}_2F_1\left(\begin{matrix} a, b \\ a-b+n+1 \end{matrix}; -1\right) = \frac{2^{n-2b}\Gamma(b-n)\Gamma(a-b+n+1)}{\Gamma(b)\Gamma(a-2b+n+1)} \times \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\Gamma(\frac{a+k+n+1}{2}-b)}{\Gamma(\frac{a+k-n+1}{2})}, \tag{1.16} $$

and

$$ {}_2F_1\left(\begin{matrix} a, b \\ a-b-n+1 \end{matrix}; -1\right) = \frac{2^{-2b-n}\Gamma(a-b-n+1)}{\Gamma(a-2b-n+1)} \times \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(\frac{a+k-n+1}{2}-b)}{\Gamma(\frac{a+k-n+1}{2})}. \tag{1.17} $$

The results (1.16) and (1.17) for $n \leq 5$ are recorded in [15].

2. Reduction formulas for Ψ_2

In this section, we obtain several reduction formulas for Ψ_2.

(1) For $c \neq 0, -1, -2 \cdots$ and $n = 0, 1, 2, \ldots$, the following result holds true:

$$ \begin{aligned} &\Psi_2(a; c, c+n; z, -z) \\ &= \frac{(-1)^n 2^{2c+n-2}\Gamma(c)}{\Gamma(2c+n)} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \\ &\times \left\{ \frac{(2c+n-1)\Gamma(c+\frac{k+n-1}{2})}{\Gamma(\frac{k-n+1}{2})} {}_4F_5\left[\begin{matrix} \frac{a}{2}, \frac{a+1}{2}, \frac{n-k+1}{2}, c+\frac{n+k-1}{2} \\ \frac{1}{2}, \frac{c+n}{2}, \frac{c+n+1}{2}, c+\frac{n-1}{2}, c+\frac{n}{2} \end{matrix}; -z^2\right] \right. \\ &\left. + \frac{4az\,\Gamma\left(c+\frac{k+n}{2}\right)}{(c+n)\Gamma\left(\frac{k-n}{2}\right)} {}_4F_5\left[\begin{matrix} \frac{a+1}{2}, \frac{a}{2}+1, \frac{n-k}{2}+1, c+\frac{n+k}{2} \\ \frac{3}{2}, \frac{c+n+1}{2}, \frac{c+n}{2}+1, c+\frac{n}{2}, c+\frac{n+1}{2} \end{matrix}; -z^2\right] \right\}. \end{aligned} \tag{2.1} $$

Proof: Setting $w = -z$ and replacing z by $-z$, and c' by $c+n$ respectively in (1.10), we get

$$\Psi_2(a;\ c,\ c+n;\ z,\ -z) = \sum_{m=0}^{\infty} \frac{(a)_m}{(c)_m}\, {}_2F_1\left[\begin{matrix} -m,\ -m-c+1 \\ c+n \end{matrix};-1\right] \frac{z^m}{m!}.$$

Using the result (1.16) and separating the series into two parts with even and odd powers of z and summing up the series with respect to m, we arrive at the right-hand side of (2.1). This completes the proof.

The result (2.1) for $n \le 5$ is recorded in [15].

In exactly the same manner, the following three general reduction formulas can be established by using the results (1.16) or (1.17).

(2) For $c, c-n \ne 0, -1, -2 \cdots$ and $n = 0, 1, 2, \ldots$, the following relation holds true:

$$\begin{aligned}
&\Psi_2(a; c, c-n; z, -z) \\
&= \frac{2^{2c-n-2}\,\Gamma(c-n)}{\Gamma(2c-n)} \sum_{k=0}^{n} \binom{n}{k} \\
&\times \left\{ \frac{(2c-n-1)\Gamma(c+\frac{k-n-1}{2})}{\Gamma(\frac{k-n+1}{2})}\, {}_4F_5\left[\begin{matrix} \frac{a}{2}, \frac{a+1}{2}, \frac{n-k+1}{2}, c+\frac{k-n-1}{2} \\ \frac{1}{2}, \frac{c}{2}, \frac{c+1}{2}, c-\frac{n}{2}, c-\frac{n+1}{2} \end{matrix}; -z^2\right] \right. \\
&\left. + \frac{4az\Gamma(c+\frac{k-n}{2})}{c\Gamma(\frac{k-n}{2})}\, {}_4F_5\left[\begin{matrix} \frac{a+1}{2}, \frac{a}{2}+1, \frac{n-k}{2}+1, c+\frac{k-n}{2} \\ \frac{3}{2}, \frac{c+1}{2}, \frac{c}{2}+1, c-\frac{n}{2}, c-\frac{n-1}{2} \end{matrix}; -z^2\right] \right\}. \qquad (2.2)
\end{aligned}$$

(3) For $c \ne 0, -1, -2 \cdots$ and $n = 0, 1, 2, \ldots$, the following relation holds true:

$$\begin{aligned}
&\Psi_2(a; c, 2-c+n; z, -z) \\
&= \frac{(-2)^n}{n!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \\
&\times \left\{ \frac{\Gamma(\frac{k-c+n}{2}+1)}{\Gamma(\frac{k-c-n}{2}+1)}\, {}_5F_6\left[\begin{matrix} 1, \frac{a}{2}, \frac{a+1}{2}, \frac{c+n-k}{2}, \frac{n-c+k}{2}+1 \\ \frac{n+1}{2}, \frac{n}{2}+1, \frac{c}{2}, \frac{c+1}{2}, \frac{n-c}{2}+1, \frac{n-c+3}{2} \end{matrix}; -z^2\right] \right. \\
&+ \frac{4az}{c(n+1)(2-c+n)} \frac{\Gamma\frac{n-c+k+3}{2})}{\Gamma(\frac{k-c-n+1}{2})} \\
&\left. \cdot\, {}_5F_6\left[\begin{matrix} 1, \frac{a+1}{2}, \frac{a}{2}+1, \frac{c+n-k+1}{2}, \frac{n+k-c+3}{2} \\ \frac{n}{2}+1, \frac{n+3}{2}, \frac{c+1}{2}, \frac{c}{2}+1, \frac{n-c+3}{2}, \frac{n-c}{2}+2 \end{matrix}; -z^2\right] \right\}. \qquad (2.3)
\end{aligned}$$

(4) For $c, 2-c-n \neq 0, -1, -2\cdots$ and $n = 0, 1, 2, \ldots$, the following relation holds true.

$$\Psi_2(a; c, 2-c-n; z, -z) = 2^{-n} \sum_{k=0}^{n} \binom{n}{k} \left\{ {}_4F_5 \left[\begin{matrix} \frac{a}{2}, \frac{a+1}{2}, \frac{c-k+n}{2}, \frac{k-n-c}{2}+1 \\ \frac{1}{2}, \frac{c}{2}, \frac{c+1}{2}, 1-\frac{c+n}{2}, \frac{3-c-n}{2} \end{matrix} ; -z^2 \right] \right.$$
$$\left. + \frac{2a(c-k+n-1)z}{c(c+n-2)} {}_4F_5 \left[\begin{matrix} \frac{a+1}{2}, \frac{a}{2}+1, \frac{c-k+n+1}{2}, \frac{k-n-c+3}{2} \\ \frac{3}{2}, \frac{c+1}{2}, \frac{c}{2}+1, \frac{3-c-n}{2}, 2-\frac{c+n}{2} \end{matrix} ; -z^2 \right] \right\}. \tag{2.4}$$

(5) Further results for Ψ_2.
Setting $c = c' = \frac{1}{2}$ in (1.10), we get

$$\Psi_2\left(a; \frac{1}{2}, \frac{1}{2}; w, z\right) = \sum_{m=0}^{\infty} \frac{(a)_k}{\left(\frac{1}{2}\right)_{k!}} {}_2F_1 \left[\begin{matrix} -k, -k+\frac{1}{2} \\ \frac{1}{2} \end{matrix} ; \frac{z}{w} \right] \frac{w^k}{k!}. \tag{2.5}$$

Using the Euler transformation

$${}_2F_1\left(\begin{matrix} \alpha, \beta \\ \gamma \end{matrix} ; z \right) = (1-z)^{-\alpha} {}_2F_1\left(\begin{matrix} \alpha, \gamma-\beta \\ \gamma \end{matrix} ; -\frac{z}{1-z} \right), \tag{2.6}$$

we have

$$\Psi_2\left(a; \frac{1}{2}, \frac{1}{2}; w, z\right) = \sum_{k=0}^{\infty} \frac{(a)_k}{\left(\frac{1}{2}\right)_k} (w-z)^k {}_2F_1 \left[\begin{matrix} -k, k \\ \frac{1}{2} \end{matrix} ; \frac{z}{z-w} \right]. \tag{2.7}$$

Due to the formula [4, (7.3.3.3), p. 486]

$${}_2F_1 \left[\begin{matrix} -a, a \\ \frac{1}{2} \end{matrix} ; -z^2 \right] = \frac{1}{2}[(\sqrt{1+z^2}+z)^{2a} + (\sqrt{1+z^2}-z)^{2a}] \tag{2.8}$$

and summing up the series, we get

$$\Psi_2\left(a; \frac{1}{2}, \frac{1}{2}; w, z\right) = \frac{1}{2}\left[{}_1F_1 \left[\begin{matrix} a \\ \frac{1}{2} \end{matrix} ; (\sqrt{w}+\sqrt{z})^2 \right] + {}_1F_1 \left[\begin{matrix} a \\ \frac{1}{2} \end{matrix} ; (\sqrt{w}-\sqrt{z})^2 \right] \right]. \tag{2.9}$$

Also, using the connection with parabolic cylinder function [4, (7.11.1.9), p. 579]

$${}_1F_1 \left[\begin{matrix} a \\ \frac{1}{2} \end{matrix} ; z \right] = \frac{2^{a-1}}{\sqrt{\pi}} \Gamma\left(a+\frac{1}{2}\right) \mathrm{e}^{z/2}[D_{-2a}(-\sqrt{2z}) + D_{-2a}(\sqrt{2z})], \tag{2.10}$$

we obtain

$$\Psi_2\left(a;\frac{1}{2},\frac{1}{2};w,z\right)=\frac{2^{a-2}\,\mathrm{e}^{(w+z)/2}}{\sqrt{\pi}}\Gamma\left(a+\frac{1}{2}\right)$$
$$\times\left\{\mathrm{e}^{\sqrt{w}\sqrt{z}}[D_{-2a}(-\sqrt{2}(\sqrt{w}+\sqrt{z}))+D_{-2a}(\sqrt{2}(\sqrt{w}+\sqrt{z}))]\right.$$
$$\left.+\,\mathrm{e}^{-\sqrt{w}\sqrt{z}}[D_{-2a}(-\sqrt{2}(\sqrt{w}-\sqrt{z}))+D_{-2a}(\sqrt{2}(\sqrt{w}-\sqrt{z}))]\right\}. \tag{2.11}$$

Further, if we put $a=-n$ in (2.11), where n is a non-negative integer and making use of the result [4, (7.11.1.19), p. 580]

$${}_1F_1\left[\begin{matrix}-n\\ \frac{1}{2}\end{matrix};z\right]=(-1)^n\frac{n!}{(2n)!}\,H_{2n}(\sqrt{z}), \tag{2.12}$$

we get

$$\Psi_2\left(-n;\frac{1}{2},\frac{1}{2};w,z\right)=\frac{(-1)^n n!}{(2n)!}[H_{2n}(\sqrt{w}+\sqrt{z})+H_{2n}(\sqrt{w}-\sqrt{z})], \tag{2.13}$$

where $H_n(z)$ is the Hermite polynomial [16, 8.2(9), p. 117] and

$$D_n(z)=2^{-n/2}\mathrm{e}^{-z^2/4}\,H_{2n}\left(\frac{z}{\sqrt{2}}\right). \tag{2.14}$$

(6) Setting $c=\frac{1}{2}$, $c'=\frac{3}{2}$ and making use of the relations

$${}_2F_1\left[\begin{matrix}a,a+\frac{1}{2}\\ \frac{3}{2}\end{matrix};z^2\right]=\frac{1}{2z(1-2a)}[(1+z)^{1-2a}-(1-z)^{1-2a}], \tag{2.15}$$

$${}_1F_1\left[\begin{matrix}a\\ \frac{3}{2}\end{matrix};z\right]=\frac{2^{a-5/2}}{\sqrt{\pi z}}\Gamma\left(a-\frac{1}{2}\right)\,e^{1/2z}[D_{1-2a}(-\sqrt{2z})-D_{1-2a}(\sqrt{2z})] \tag{2.16}$$

and

$${}_1F_1\left[\begin{matrix}-n\\ \frac{3}{2}\end{matrix};z\right]=\frac{(-1)^n\,n!}{2(2n+1)!\,\sqrt{z}}H_{2n+1}(\sqrt{z}), \tag{2.17}$$

we obtain the following representations:

$$\Psi_2\left(a;\frac{1}{2},\frac{3}{2};w,z\right)$$
$$=\frac{2^{a-7/2}\,\mathrm{e}^{(1/2)(w+z)}}{\sqrt{\pi\,z}}\Gamma\left(a-\frac{1}{2}\right)$$
$$\times\{\mathrm{e}^{\sqrt{w}\sqrt{z}}[D_{1-2a}(-\sqrt{2}(\sqrt{w}+\sqrt{z}))-D_{1-2a}(\sqrt{2}(\sqrt{w}+\sqrt{z}))]$$
$$-\mathrm{e}^{-\sqrt{w}\sqrt{z}}[D_{1-2a}(-\sqrt{2}(\sqrt{w}-\sqrt{z}))-D_{1-2a}(\sqrt{2}(\sqrt{w}-\sqrt{z}))]\}, \tag{2.18}$$

$$\Psi_2\left(-n;\frac{1}{2},\frac{3}{2};w,z\right)=\frac{(-1)^n\,n!}{4(2n+1)!\,\sqrt{z}}[H_{2n+1}(\sqrt{w}+\sqrt{z})-H_{2n+1}(\sqrt{w}-\sqrt{z})]. \quad (2.19)$$

■

3. Reduction formulas for Φ_2

In this section, we establish several reduction formulas for Φ_2.

(1) For $c \neq 0,-1,-2\cdots$ and $n=0,1,2,\ldots$, the following equality holds true:

$$\begin{aligned}&\Phi_2(b,b+n;c;z,-z)\\&=\frac{2^{-2b-n}\Gamma(1-b)}{(b)_n\,\Gamma(1-2b-n)}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\\&\quad\times\left\{\frac{\Gamma(\frac{k-n+1}{2}-b)}{\Gamma(\frac{k-n+1}{2})}\,{}_3F_4\left[\begin{matrix}\frac{n+1}{2}+b,\frac{n}{2}+b,\frac{n-k+1}{2}\\ \frac{1}{2},\frac{c}{2},\frac{c+1}{2},\frac{n-k+1}{2}+b\end{matrix};\frac{z^2}{4}\right]\right.\\&\quad\left.+\frac{(2b+n)z}{c}\frac{\Gamma(\frac{k-n}{2}-b)}{\Gamma(\frac{k-n}{2})}\,{}_3F_4\left[\begin{matrix}\frac{n+1}{2}+b,\frac{n}{2}+b+1,\frac{n-k}{2}+1\\ \frac{3}{2},\frac{c+1}{2},\frac{c}{2}+1,\frac{n-k}{2}+b+1\end{matrix};\frac{z^2}{4}\right]\right\}. \quad (3.1)\end{aligned}$$

Proof: Replacing b' by $b+n$, z by $-z$ and w by z in (1.11), we have

$$\Phi_2(b,b+n;c;z,-z)=\sum_{m=0}^{\infty}\frac{(b)_m}{(c)_m}\,{}_2F_1\left[\begin{matrix}-m,b+n\\1-b-m\end{matrix};-1\right]\frac{z^m}{m!}.$$

Using the result (1.16) and separating the series into two parts with even and odd powers of z and finally summing up the series with respect to m, we arrive at the right-hand side of (3.1). The result (3.1) for $n \leq 5$ is recorded in [15]. In exactly the same manner, the following result can be established by using the relation (1.17).

(2) For $c \neq 0,-1,-2\cdots$ and $n=0,1,2,\ldots$, the following formula holds true:

$$\begin{aligned}&\Phi_2(b,b-n;c;z,-z)\\&=(-1)^n\frac{2^{-2b+n}\Gamma(1-b)}{\Gamma(1-2b+n)}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\\&\quad\times\left\{\frac{\Gamma(\frac{n+k+1}{2}-b)}{\Gamma(\frac{k-n+1}{2})}\,{}_3F_4\left[\begin{matrix}b-\frac{n}{2},b-\frac{n-1}{2},\frac{n-k+1}{2}\\ \frac{1}{2},\frac{c}{2},\frac{c+1}{2},\frac{1-k-n}{2}+b\end{matrix};\frac{z^2}{4}\right]\right.\\&\quad\left.-\frac{z(2b-n)}{c}\frac{\Gamma(\frac{k+n}{2}-b)}{\Gamma(\frac{k-n}{2})}\,{}_3F_4\left[\begin{matrix}\frac{1-n}{2}+b,1-\frac{n}{2}+b,1-\frac{k-n}{2}\\ \frac{3}{2},\frac{c+1}{2},\frac{c}{2}+1,1+b-\frac{n+k}{2}\end{matrix};\frac{z^2}{4}\right]\right\}. \quad (3.2)\end{aligned}$$

(3) A reduction formula for $\Phi_2(b,b';c+r;w,z)$ can be derived using the integral representation [10, (4.1), p. 8]

$$\Phi_2(b,b';c+r;w,z)=\frac{1}{B(c,r)}\int_0^1 t^{c-1}(1-t)^{r-1}\,{}_1F_1\left[\begin{matrix}b\\c\end{matrix};wt\right]{}_1F_1\left[\begin{matrix}b'\\r\end{matrix};z(1-t)\right]\mathrm{d}t. \quad (3.3)$$

Posing $b'=r=1$, we get

$$\Phi_2(b,1;c+1;w,z)=ce^z\int_0^1 t^{c-1}e^{-tz}{}_1F_1\left[\begin{matrix}b\\c\end{matrix};wt\right]\mathrm{d}t. \quad (3.4)$$

Application of the differentiation formula [7, (4.13), p. 798]

$$D_z^n[z^{b'+n-1}\Phi_2(b,b';c;w,z)]=(b')_n z^{b'-1}\Phi_2(b,b'+n;c;w,z) \quad (3.5)$$

with $b'=1$ and $c\to c-1$ yields

$$D_z^n[z^n\Phi_2(b,1;c;w,z)]=n!\Phi_2(b,n+1;c;w,z). \quad (3.6)$$

Thus, using the formula

$$D_z^n[z^\lambda\, e^{az}]=n!z^{\lambda-n}e^{az}L_n^{\lambda-n}(-az), \quad (3.7)$$

we get

$$\Phi_2(b,n+1;c;w,z)=(c-1)e^z\sum_{k=0}^n\frac{(-1)^k}{k!}z^kL_{n-k}^k(-z)\int_0^1 t^{c+k-2}e^{-tz}\,{}_1F_1\left[\begin{matrix}b\\c-1\end{matrix};wt\right]\mathrm{d}t. \quad (3.8)$$

If we take $c=2b+1$ and use the second Kummer formula [4, (7.11.1.5), p. 579]

$${}_1F_1\left[\begin{matrix}b\\2b\end{matrix};z\right]==\mathrm{e}^{z/2}\Gamma\left(b+\frac{1}{2}\right)\left(\frac{z}{4}\right)^{1/2-b}\;I_{b-1/2}\left(\frac{z}{2}\right),$$

we get

$$\Phi_2(b,n+1;2b+1;w,z)=2^{2b}b\;w^{1/2-b}e^z\Gamma\left(b+\frac{1}{2}\right)\sum_{k=0}^n\frac{(-z)^k}{k!}\,L_{n-k}^k(z)$$
$$\cdot\int_0^1 t^{b+k-\frac{1}{2}}\,e^{(w-2z)t/2}\,I_{b-1/2}\left(\frac{wt}{2}\right)\mathrm{d}t. \quad (3.9)$$

For $w=z$, we can evaluate the integral and obtain the following reduction formula:

$$\Phi_2(b,n+1;2b+1;z,z)=2\,b\,e^z\sum_{k=0}^n\frac{(-z)^k}{k!(k+2b)}\,L_{n-k}^k(-z)\,{}_2F_2\left[\begin{matrix}b,k+2b\\2b,k+2b+1\end{matrix};-z\right]. \quad (3.10)$$

In exactly the similar way, we can obtain the reduction formula

$$\Phi_2(n+1,b;2b+1;z,2z)=2\,b\,e^z\sum_{k=0}^n\frac{(-z)^k}{k!(k+2b)}\,L_{n-k}^k(-z)\,{}_1F_2\left[\begin{matrix}b+\frac{k}{2}\\b+\frac{1}{2},b+\frac{k}{2}+1\end{matrix};\frac{z^2}{4}\right]. \quad (3.11)$$

(4) A reduction formula for $\Phi_2(b,b';b+n;z,z)$ can be obtained as follows:

If we take $c=b, r=n, w=z$ for $n=1,2,3,\ldots$ in the integral representation (3.3), we get

$$\Phi_2(b,b';b+n;z,z)=\frac{(b)_n}{(n-1)!}\int_0^1 t^{b-1}(1-t)^{n-1}e^{tz}\,{}_1F_1\left[\begin{matrix}b'\\ n\end{matrix};z(1-t)\right]\mathrm{d}t. \tag{3.12}$$

Evaluating the integral, we obtain the representation

$$\Phi_2(b,b';b+n;z,z)=\frac{(b)_n e^z}{(n-1)!}\sum_{k=0}^{n-1}\frac{(-1)^k}{k+b}\binom{n-1}{k}\,{}_2F_2\left[\begin{matrix}1,n-b'\\ n,k+b+1\end{matrix};-z\right]. \tag{3.13}$$

■

4. Reduction formulas for Φ_3

In this section, we establish two general reduction formulas for Φ_3.

(1) The following result holds true:

$$\begin{aligned}\Phi_3&\left(b;\frac{b-n}{2}+1;z,-z^2\right)\\ &=2^{-n}\sum_{k=0}^{n}\binom{n}{k}\left\{{}_2F_3\left[\begin{matrix}\frac{b-n+2k+2}{4},\frac{2-2k-b+n}{4}\\ \frac{1}{2},\frac{b-n+2}{4},\frac{b-n}{4}+1\end{matrix};-z^2\right]\right.\\ &\left.+\frac{2(b-n+2k)\,z}{b-n+2}\,{}_2F_3\left[\begin{matrix}\frac{b-n+2k}{4}+1,1-\frac{b-n+2k}{4}\\ \frac{3}{2},\frac{b-n}{4}+1,\frac{b-n+6}{4}\end{matrix};-z^2\right]\right\}.\end{aligned} \tag{4.1}$$

for $n=0,1,2,\ldots$

Proof: Replacing z by $-z^2$ in (1.12) and posing $w=z$, $c=(b-n)/2+1$, we get

$$\Phi_3\left(b;\frac{b-n}{2}+1;z,-z^2\right)=\sum_{m=0}^{\infty}\frac{z^m}{m!}\,{}_2F_1\left[\begin{matrix}-m,\frac{b+n}{2}-m\\ \frac{b-n}{2}+1\end{matrix};-1\right].$$

Using the result (1.17), separating the series into two parts with even and odd powers of z and summing up the series with respect to m, we arrive at the right-hand side of (4.1).

In exactly the same manner, a similar result can also be established with the help of the result (1.16).

(2) The following result holds true:

$$\Phi_3\left(b;\frac{b+n}{2}+1;z,-z^2\right)$$
$$=\frac{(-2)^n}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left\{\frac{\Gamma(\frac{b+n+2k+2}{4})}{\Gamma(\frac{b-3n+2k+2}{4})}\right.$$
$$\cdot{}_3F_4\left[\begin{matrix}1,\frac{b+n+2k+2}{4},\frac{2-2k-b+3n}{4}\\ \frac{n+1}{2},\frac{n}{2}+1,\frac{b+n+2}{4},\frac{b+n}{4}+1\end{matrix};-z^2\right]$$
$$+\frac{8z}{(n+1)(b+n+2)}\frac{\Gamma(\frac{b+n+2k}{4}+1)}{\Gamma(\frac{b-3n+2k}{4})}$$
$$\left.\cdot\,{}_3F_4\left[\begin{matrix}1,\frac{b+n+2k}{4}+1,1-\frac{b-3n+2k}{4}\\ \frac{n}{2}+1,\frac{n+3}{2},\frac{b+n}{4}+1,\frac{b+n+6}{4}\end{matrix};-z^2\right]\right\} \tag{4.2}$$

for $n=0,1,2,\ldots$. If $n=0$, the representations (4.1) or (4.2) yield the formula:

$$\Phi_3(b;\frac{b}{2}+1;z,-z^2)={}_1F_2\left[\begin{matrix}\frac{2-b}{4}\\ \frac{1}{2},\frac{b}{4}+1\end{matrix};-z^2\right]+\frac{2bz}{b+2}\,{}_1F_2\left[\begin{matrix}1-\frac{b}{4}\\ \frac{3}{2},\frac{b+6}{4}\end{matrix};-z^2\right] \tag{4.3}$$

■

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