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A NOTE ON GENERALIZATIONS OF TWO RESULTS DUE TO BAILEY

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Abstract

Elementary proofs of generalizations of two results, due to Bailey, involving the product of generalized hypergeometric functions, are provided.

1. Introduction

Bailey [1] established a large number of very interesting results involving the product of generalized hypergeometric functions by employing

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classical summation theorems such as those of Gauss, Gauss second, Kummer and Bailey for the series ${}_2F_1$, Watson, Dixon, Whipple and Saalschütz for the series ${}_3F_2$. For the details of the generalized hypergeometric functions ${}_pF_q$, we refer, for example, to [5] and [8, Section 1.5].

We recall the following two results due to Bailey [1]:

$$(1-x)^{-2\alpha}\ {}_2F_1\left[\begin{matrix}\alpha,\ \alpha+1/2; \\ 1/2;\end{matrix}\ -\frac{x^2}{(1-x)^2}\right] = \sum_{n=0}^{\infty} 2^{n/2}(2\alpha)_n \cos\left(\frac{n\pi}{4}\right)\frac{x^n}{n!} \quad (1.1)$$

and

$$(1-x)^{-2\alpha}\ {}_2F_1\left[\begin{matrix}\alpha,\ \alpha+1/2; \\ 3/2;\end{matrix}\ -\frac{x^2}{(1-x)^2}\right]$$

$$= \sum_{n=0}^{\infty} 2^{(n+1)/2}(2\alpha)_n \sin\left(\frac{(n+1)\pi}{4}\right)\frac{x^n}{(n+1)!}, \quad (1.2)$$

where $(\lambda)_\nu$ denotes the Pochhammer symbol defined (for $\lambda, \nu \in \mathbb{C}$), in terms of the familiar Gamma function Γ, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C}\setminus\{0\}), \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}). \end{cases} \quad (1.3)$$

Here and in the following, let $\mathbb{C}, \mathbb{R}^+, \mathbb{R}^-, \mathbb{Z}$ and $\mathbb{N}$ be the sets of complex numbers, positive real numbers, negative real numbers, integers, and positive integers, respectively, and, let

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} \text{ and } \mathbb{Z}_0^- := \mathbb{Z}\setminus\mathbb{N}.$$

It is interesting to observe that, in (1.1) and (1.2), if we replace x by $\frac{x}{\alpha}$ and let $\alpha \to \infty$, after a little simplification, we get the following results which are also due to [1]:

$$e^x\ {}_0F_1\left[\begin{matrix} —; \\ 1/2; \end{matrix} -\frac{x^2}{4}\right] = \sum_{n=0}^{\infty} 2^{n/2} \cos\left(\frac{n\pi}{4}\right)\frac{x^n}{n!} \tag{1.4}$$

and

$$e^x\ {}_0F_1\left[\begin{matrix} —; \\ 3/2; \end{matrix} -\frac{x^2}{4}\right] = \sum_{n=1}^{\infty} 2^{n/2} \sin\left(\frac{n\pi}{4}\right)\frac{x^{n-1}}{n!}. \tag{1.5}$$

The results (1.1), (1.2), (1.4) and (1.5) were established by Bailey [1] who used the following classical Kummer's summation theorem (see, e.g., [5, 8]):

$$_2F_1\left[\begin{matrix} a, b; \\ 1+a-b; \end{matrix} -1\right] = \frac{\Gamma\left(1+\frac{1}{2}a\right)\Gamma(1+a-b)}{\Gamma(1+a)\Gamma\left(1+\frac{1}{2}a-b\right)} \tag{1.6}$$

$$(\Re(b) < 1;\ 1+a-b \in \mathbb{C}\setminus\mathbb{Z}_0^-).$$

Choi and Rathie [2] presented the following generalizations of Bailey's results (1.1), (1.2), (1.4) and (1.5):

$$(1-x)^{-2\alpha}\ {}_2F_1\left[\begin{matrix} \alpha, \alpha+1/2; \\ 1/2; \end{matrix} -\frac{b^2x^2}{a^2(1-x)^2}\right]$$

$$= \sum_{n=0}^{\infty} (2\alpha)_n \cos(n\theta)\frac{(a^2+b^2)^{n/2}x^n}{a^n n!}, \tag{1.7}$$

$$(1-x)^{-2\alpha}\ {}_2F_1\left[\begin{matrix} \alpha, \alpha+1/2; \\ 3/2; \end{matrix} -\frac{b^2x^2}{a^2(1-x)^2}\right]$$

$$= \sum_{n=0}^{\infty} (2\alpha)_n \sin((n+1)\theta)\frac{(a^2+b^2)^{(n+1)/2}x^n}{a^n bn!}, \tag{1.8}$$

$$e^x\ {}_0F_1\left[\begin{matrix} —; \\ 1/2; \end{matrix} -\frac{b^2x^2}{a^2}\right] = \sum_{n=0}^{\infty} \cos(n\theta)(a^2+b^2)^{n/2}\frac{x^n}{a^n n!} \tag{1.9}$$

and

$$e^x\ {}_0F_1\left[\begin{matrix} \text{—}; \\ 3/2; \end{matrix}\ -\frac{b^2x^2}{a^2}\right] = \sum_{n=1}^{\infty} \sin(n\theta)\,(a^2+b^2)^{n/2}\,\frac{x^{n-1}}{a^{n-1}bn!}, \tag{1.10}$$

where θ is given, here and in what follows, by

$$\theta := \begin{cases} \arctan\left(\dfrac{b}{a}\right) & (a,\ b \in \mathbb{R}^+), \\ \pi - \arctan\left(\dfrac{b}{|a|}\right) & (a \in \mathbb{R}^-;\ b \in \mathbb{R}^+), \\ \arctan\left(\dfrac{|b|}{|a|}\right) - \pi & (a,\ b \in \mathbb{R}^-), \\ -\arctan\left(\dfrac{|b|}{a}\right) & (a \in \mathbb{R}^+;\ b \in \mathbb{R}^-). \end{cases} \tag{1.11}$$

They [2] obtained the results (1.7)-(1.10) by using the following two summation formulas due to Qureshi et al. [4]:

$${}_2F_1\left[\begin{matrix} -\frac{n}{2},\ \frac{1-n}{2}; \\ 1/2; \end{matrix}\ -\frac{b^2}{a^2}\right] = \frac{(a^2+b^2)^{n/2}}{a^n}\cos(n\theta) \tag{1.12}$$

$$\left(n\theta \neq \frac{2k+1}{2}\pi\ (k \in \mathbb{Z})\right)$$

and

$${}_2F_1\left[\begin{matrix} -\frac{n}{2},\ \frac{1-n}{2}; \\ 3/2; \end{matrix}\ -\frac{b^2}{a^2}\right] = \frac{(a^2+b^2)^{(n+1)/2}}{(n+1)a^n b}\sin((n+1)\theta) \tag{1.13}$$

$$((n+1)\theta \neq k\pi\ (k \in \mathbb{Z})).$$

Also, Choi and Rathie [2] proved (1.12) and (1.13) in an elementary way. Choi and Rathie [3] showed (1.9) and (1.10) in an elementary way without using (1.12) and (1.13).

Qureshi et al. [4] presented the following extensions of (1.9) and (1.10):

$$e^{ax}\cos(bx+c)=\sum_{n=0}^{\infty}\cos(n\theta+c)\frac{(x\sqrt{a^2+b^2})^n}{n!} \tag{1.14}$$

$$\left(n\theta+c\neq\frac{2k+1}{2}\pi\ (k\in\mathbb{Z},\ n\in\mathbb{N}_0)\right)$$

and

$$e^{ax}\sin(bx+c)=\sum_{n=0}^{\infty}\sin(n\theta+c)\frac{(x\sqrt{a^2+b^2})^n}{n!} \tag{1.15}$$

$$(n\theta+c\neq k\pi\ (k\in\mathbb{Z},\ n\in\mathbb{N}_0)).$$

Sukanya et al. [6] established (1.14) and (1.15) in a very elementary way. Sukanya et al. [7] also proved (1.1) and (1.2) in a very elementary way without using Kummer's summation theorem (1.6).

Here, we aim to prove the results (1.7) and (1.8) in an elementary way without using the summation formulas (1.12) and (1.13).

2. Derivations of (1.7) and (1.8)

We establish the identity (1.7). To do this, let $\mathcal{L}$ be the right-hand side of (1.7). First, we assume that x and α are real,

$$\mathcal{L}=\Re\left\{\sum_{n=0}^{\infty}\frac{(2\alpha)_n}{n!}\left(\frac{x}{a}\sqrt{a^2+b^2}\,e^{i\theta}\right)^n\right\}$$

$$=\Re\left\{\sum_{n=0}^{\infty}\frac{(2\alpha)_n}{n!}\left(\frac{x(a+ib)}{a}\right)^n\right\}$$

$$=\Re\left\{{}_1F_0\left[\begin{matrix}2\alpha; & \frac{x(a+ib)}{a}\\ —; & \end{matrix}\right]\right\}.$$

By using the binomial theorem (see, e.g., [8, p. 67, equation (22)]):

$$(1-z)^{-\alpha} = \sum_{n=0}^{\infty} (\alpha)_n \frac{z^n}{n!} \qquad (\alpha \in \mathbb{C};\ |z| < 1),$$

we have

$$\mathcal{L} = \Re\left\{\left(1 - \frac{x(a+ib)}{a}\right)^{-2\alpha}\right\}$$

$$= \Re\left\{\left((1-x) - i\frac{b}{a}x\right)^{-2\alpha}\right\}.$$

Then we get

$$\mathcal{L} = (1-x)^{-2\alpha}\Re\left\{\left(1 - \frac{ibx}{a(1-x)}\right)^{-2\alpha}\right\}$$

$$= (1-x)^{-2\alpha}\Re\left\{{}_1F_0\left[\begin{matrix}2\alpha; \\ —;\end{matrix}\ \frac{ibx}{a(1-x)}\right]\right\}$$

$$= (1-x)^{-2\alpha}\Re\left\{\sum_{n=0}^{\infty}\frac{(2\alpha)_n}{n!}\left(\frac{ibx}{a(1-x)}\right)^n\right\}.$$

We have

$$\mathcal{L} = (1-x)^{-2\alpha}\sum_{n=0}^{\infty}(-1)^n\,\frac{(2\alpha)_{2n}}{(2n)!}\left(\frac{bx}{a(1-x)}\right)^{2n}.$$

Using

$$(2\alpha)_{2n} = 2^{2n}(\alpha)_n(\alpha+1/2)_n \text{ and } (2n)! = 2^{2n}(1/2)_n n!,$$

we obtain

$$\mathcal{L} = (1-x)^{-2\alpha}\sum_{n=0}^{\infty}\frac{(\alpha)_n(\alpha+1/2)_n}{(1/2)_n n!}\left(-\frac{b^2x^2}{a^2(1-x)^2}\right)^n,$$

which, upon using the notation ${}_2F_1$, is equal to the left-hand side of (1.7).

When x and α are complex numbers, by the principle of analytic continuation, (1.7) holds for suitable complex numbers x and α. This completes the proof of (1.7).

The proof of (1.8) would run parallel to that of (1.7). So, its detailed account is left to the interested reader.

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