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A NEW CLASS OF INTEGRALS INVOLVING GENERALIZED HYPERGEOMETRIC FUNCTIONS

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Abstract

The objective of this paper is to evaluate a general class of integrals involving generalized hypergeometric functions which contains 25 integrals. The results are derived with the help of generalized Watson's summation theorem due to Lavoie et al. [5]. Fifty presumably new and interesting integrals have also been obtained as special cases of our main findings.

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1. Introduction and Preliminaries

The natural generalization of the Gauss's hypergeometric function ${}_2F_1$ is called the *generalized hypergeometric series* ${}_pF_q$ $(p, q \in \mathbb{N}_0)$ defined by (see, e.g. [1], [6, p. 73] and [7, pp. 71-75]):

$$
{}_pF_q\left[\begin{matrix}\alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q;\end{matrix}\ z\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!}
$$

$$
= {}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z), \tag{1}
$$

where $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by (see [7, p. 2 and p. 5]):

$$
(\lambda)_n := \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C}\setminus\mathbb{Z}_0^-)
$$

$$
= \begin{cases} 1 & (n = 0) \\ \lambda(\lambda + 1)\ldots(\lambda + n - 1) & (n \in \mathbb{N}) \end{cases} \tag{2}
$$

and $\Gamma(\lambda)$ is the familiar gamma function. Here, an empty product is interpreted as 1, and we assume (for simplicity) that the variable z, the numerator parameters $\alpha_1, \ldots, \alpha_p$, and the denominator parameters $\beta_1, \ldots, \beta_q$, take on complex values, provided that no zeros appear in the denominator of (1), that is,

$$
(\beta_j \in \mathbb{C}\setminus\mathbb{Z}_0^-, \ j = 1, \ldots, q). \tag{3}
$$

Here, and in the following, let $\mathbb{C}$, $\mathbb{Z}$ and $\mathbb{N}$ be the sets of complex numbers, integers and positive integers, respectively, and let

$$
\mathbb{N}_0 := \mathbb{N} \cup \{0\} \quad \text{and} \quad \mathbb{Z}_0^- := \mathbb{Z}\setminus\mathbb{N}.
$$

For more details of ${}_pF_q$ including its convergence, its various special and limiting cases, and its further diverse generalizations, one may be referred, for example, to [1, 6, 7].

It is worthy of note that whenever the generalized hypergeometric function ${}_pF_q$ (including ${}_2F_1$) with its specified argument (for example, unit argument) can be summed to be expressed in terms of the gamma functions, the result may be very important from both theoretical and applicable points of view. Here, the classical summation theorems for the hypergeometric series ${}_2F_1$ such as those of Gauss and Gauss second, Kummer, and Bailey; Watson's, Dixon's, Whipple's and Saalschütz's summation theorems for the series ${}_3F_2$ and others play important roles in theory and application. During 1992-1996, in a series of works [3-5], Lavoie et al. have generalized the above mentioned classical summation theorems for ${}_3F_2$ of Watson, Dixon, and Whipple and presented a large number of special and limiting cases of their results. Those results have also been obtained and verified with the help of computer programs (for example, Mathematica).

For our present investigation, we recall the following classical Watson's summation theorem (see, e.g., [1, 6]; see also [7, p. 351]):

$$
{}_3F_2\left[\begin{matrix} a,\ b,\ c; \\ \frac{1}{2}(a+b+1),\ 2c; \end{matrix}\ 1\right]
$$
$$
= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)}, \tag{4}
$$

provided $\Re(2c-a-b) > -1$.

Lavoie et al. [3] established a generalization of (4), which contains twenty five identities and is recorded in the following single form:

$$
{}_3F_2\left[\begin{matrix} a,\ b,\ c; \\ \frac{1}{2}(a+b+i+1),\ 2c+j; \end{matrix}\ 1\right]
$$
$$
= \mathcal{A}_{j,i}\, 2^{a+b+i-2}\, \frac{\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2}\right)\Gamma\left(c+[j/2]+\frac{1}{2}\right) \times \Gamma\left(c-\frac{1}{2}(a+b+|i+j|-j-1)\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(a)\Gamma(b)}
$$

$$\times \left\{ \mathcal{B}_{j,i} \frac{\Gamma\left(\frac{1}{2}a + \frac{1}{4}(1-(-1)^i)\right)\Gamma\left(\frac{1}{2}b\right)}{\Gamma\left(c - \frac{1}{2}a + \frac{1}{2} + [j/2] - \frac{1}{4}(-1)^j(1-(-1)^i)\right) \times \Gamma\left(c - \frac{1}{2}a + \frac{1}{2} + [j/2]\right)} \right.$$

$$\left. + \mathcal{C}_{j,i} \frac{\Gamma\left(\frac{1}{2}a + \frac{1}{4}(1+(-1)^i)\right)\Gamma\left(\frac{1}{2}b + \frac{1}{2}\right)}{\Gamma\left(c - frac12a + [(j+1)/2] + \frac{1}{4}(-1)^j(1-(-1)^i)\right) \times \Gamma\left(c - \frac{1}{2}b + [(j+1)/2]\right)} \right\}$$

$$:= \Omega \tag{5}$$

for $i, j = 0, \pm 1, \pm 2$. Here, $[x]$ denotes the greatest integer less than or equal to x and $|x|$ is the absolute value of x. The coefficients $\mathcal{A}_{j,i}$, $\mathcal{B}_{j,i}$ and $\mathcal{C}_{j,i}$ are given in the Tables 1-3.

Here, in this paper, we aim to evaluate a class of presumably new and potentially useful integrals associated with generalized hypergeometric functions:

$$\int_0^1 x^{c-1}(1-x)^{c+\ell} \, {}_3F_2\left[\begin{matrix} a, b, 2c+\ell+1; \\ \frac{1}{2}(a+b+i+1), 2c+j; \end{matrix} \; x\right] dx$$

$$(\ell \in \mathbb{Z} \text{ and } i, j = 0, \pm 1, \pm 2)$$

by mainly using the generalized Watson's summation theorem due to Lavoie et al. [5]. Fifty interesting general integrals are also given as special cases of the main result.

2. General Integral Formula

Here, we present a class of integral formulas involving the generalized hypergeometric functions ${}_3F_2$, which is asserted by the following theorem:

Theorem 1. *The following general integral formula holds*:

$$\int_0^1 x^{c-1}(1-x)^{c+\ell}\, {}_3F_2\left[\begin{matrix} a,\ b,\ 2c+\ell+1; \\ \frac{1}{2}(a+b+i+1),\ 2c+j; \end{matrix}\ x\right] dx = \frac{\Gamma(c)\Gamma(c+\ell+1)}{\Gamma(2c+\ell+1)}\,\Omega, \tag{6}$$

where Ω *is given in* (5), $\ell \in \mathbb{Z}$, *and* $i, j = 0, \pm 1, \pm 2$, *and provided* $\Re(c+\ell) > 0 \ \ (\ell \in \mathbb{N}_0)$, $\Re(c) > -\ell$, $(\ell \in \mathbb{Z}\backslash\mathbb{N}_0)$ *and* $\Re(2c - a - b + i + 2j + 1) > 0\,(i,\ j = 0, \pm 1, \pm 2)$.

Proof. Let $\mathcal{L}$ be the left side of (6). Expressing the ${}_3F_2$ in (6) as the corresponding summation in (1) and interchanging the order of integral and summation, which is guaranteed by the uniform convergence of the series on the interval, we obtain

$$\mathcal{L} = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(2c+\ell+1)_n}{\left(\frac{1}{2}(a+b+i+1)\right)_n (2c+j)_n\, n!} \int_0^1 x^{c+n-1}(1-x)^{c+\ell}\, dx.$$

Evaluating the beta integral, after a little simplification, we get

$$\mathcal{L} = \frac{\Gamma(c)\Gamma(c+\ell+1)}{\Gamma(2c+\ell+1)} \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c)_n}{\left(\frac{1}{2}(a+b+i+1)\right)_n (2c+j)_n\, n!}.$$

Using (1), we have

$$\mathcal{L} = \frac{\Gamma(c)\Gamma(c+\ell+1)}{\Gamma(2c+\ell+1)}\, {}_3F_2\left[\begin{matrix} a,\ b,\ c; \\ \frac{1}{2}(a+b+i+1),\ 2c+j; \end{matrix}\ 1\right]$$

which, upon evaluating with the aid of (5), is led to the right side of (6). This completes the proof. □

3. Special Cases

Here, as special cases of the main result (6), we present fifty interesting integral formulas, which are given in the following two corollaries. In fact, in (6), for $n \in \mathbb{N}$, letting $b = -2n$ and replacing a by $a + 2n$, or, letting $b = -2n - 1$ and replacing a by $a + 2n + 1$, we find that, in each case, one of the two terms appearing on the right sides of (6) will vanish. Then, under the given conditions, it is easy to get the following fifty desired integral formulas:

Corollary 1. *The following integral formula holds*:

$$\int_0^1 x^{c-1}(1-x)^{c+\ell}\, {}_3F_2\left[\begin{matrix} -2n,\ a+2n+2c+\ell+1; \\ \frac{1}{2}(a+i+1),\ 2c+j; \end{matrix}\ x\right] dx$$

$$= \mathcal{D}_{i,j}\,\frac{\Gamma(c)\Gamma(c+\ell+1)}{\Gamma(2c+\ell+1)}\,\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}a - c + \frac{3}{4} - \frac{(-1)^i}{4} - \left[\frac{j}{2} + \frac{1}{4}(1-(-1)^i)\right]\right)_n}{\left(c + \frac{1}{2} + \left[\frac{j}{2}\right]\right)_n\left(\frac{1}{2}a + \frac{1}{4}(1+(-1)^i)\right)_n}$$

$$:= \Omega_1 \quad (n \in \mathbb{N}_0;\ \ell \in \mathbb{Z};\ i,\ j = 0, \pm 1, \pm 2), \tag{7}$$

where $\mathcal{D}_{i,j}$ *are given in Table* 4.

Corollary 2. *The following integral formula holds*:

$$\int_0^1 x^{c-1}(1-x)^{c+\ell}\, {}_3F_2\left[\begin{matrix} -2n-1,\ a+2n+1,\ 2c+\ell+1; \\ \frac{1}{2}(a+i+1),\ 2c+j; \end{matrix}\ x\right] dx$$

$$= \mathcal{E}_{i,j}\,\frac{\Gamma(c)\Gamma(c+\ell+1)}{\Gamma(2c+\ell+1)}\,\frac{\left(\frac{3}{2}\right)_n\left(\frac{1}{2}a - c + \frac{5}{4} + \frac{(-1)^i}{4} - \left[\frac{j}{2} + \frac{1}{4}(1+(-1)^i)\right]\right)_n}{\left(c + \frac{1}{2} + \left[\frac{j+1}{2}\right]\right)_n\left(\frac{1}{2}a + \frac{1}{4}(3-(-1)^i)\right)_n}$$

$$:= \Omega_2 \quad (n \in \mathbb{N}_0;\ \ell \in \mathbb{Z};\ i,\ j = 0, \pm 1, \pm 2), \tag{8}$$

where $\mathcal{E}_{i,j}$ *are given in Table* 5.

We conclude this paper by giving further special cases of (7) and (8). Setting $i = j = 0$ in (7) and (8) yields the following integral formulas:

$$\int_0^1 x^{c-1}(1-x)^{c+\ell} {}_3F_2\left[\begin{matrix} -2n,\ a+2n,\ 2c+\ell+1; \\ \frac{1}{2}(a+1),\ 2c; \end{matrix}\ x\right] dx$$

$$= \frac{\Gamma(c)\Gamma(c+\ell+1)}{\Gamma(2c+\ell+1)} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}a - c + \frac{1}{2}\right)_n}{\left(c+\frac{1}{2}\right)_n \left(\frac{1}{2}a + \frac{1}{2}\right)_n} \quad (n \in \mathbb{N}_0;\ \ell \in \mathbb{Z}) \tag{9}$$

and

$$\int_0^1 x^{c-1}(1-x)^{c+\ell} {}_3F_2\left[\begin{matrix} -2n-1,\ a+2n+1,\ 2c+\ell+1; \\ \frac{1}{2}(a+1),\ 2c; \end{matrix}\ x\right] dx = 0$$

$$(n \in \mathbb{N}_0;\ \ell \in \mathbb{Z}). \tag{10}$$

Remark 1. The result (10) is interesting as it can be seen that for all $\ell \in \mathbb{Z}$, the value of the double integral is zero.

Table 1. Table for $\mathcal{A}_{j,i}$

$j \backslash i$	2	1	0	–1	–2
–2	$\frac{1}{2(c-1)(a-b-1)(a-b+1)}$	$\frac{1}{(c-1)(a-b)}$	$\frac{1}{2(c-1)}$	$\frac{1}{c-1}$	$\frac{1}{2(c-1)}$
–1	$\frac{1}{2(a-b-1)(a-b+1)}$	$\frac{1}{a-b}$	1	1	1
0	$\frac{1}{4(a-b-1)(a-b+1)}$	$\frac{1}{a-b}$	1	2	1
1	$\frac{1}{4(a-b-1)(a-b+1)}$	$\frac{1}{2(a-b)}$	1	2	2
2	$\frac{1}{8(c+1)(a-b-1)(a-b+1)}$	$\frac{1}{2(c+1)(a-b)}$	$\frac{1}{2(c+1)}$	$\frac{2}{c+1}$	$\frac{2}{c+1}$

Table 2. Table for $\mathcal{B}_{j,i}$

$j\backslash i$	2	1	0	-1	-2
-2	$\mathcal{B}_{-2,2}$	$c-b-1$	$\mathcal{B}_{-2,0}$	$\mathcal{B}_{-2,-1}$	$\mathcal{B}_{-2,-2}$
-1	$a-b+1$	1	1	$2c-a+b-2$	$\mathcal{B}_{-1,-2}$
0	$\mathcal{B}_{0,2}$	1	1	1	$\mathcal{B}_{0,-2}$
1	$\mathcal{B}_{1,2}$	$2c-a+b$	1	1	$a+b-1$
2	$\mathcal{B}_{2,2}$	$\mathcal{B}_{2,1}$	$\mathcal{B}_{2,0}$	$c-b+1$	$\mathcal{B}_{2,-2}$

Here,

$$\mathcal{B}_{-2,2} := c(a+b-1)-(a+1)(b+1)+2,$$

$$\mathcal{B}_{-2,0} := (c-a-1)(c-b-1)+(c-1)(c-2),$$

$$\mathcal{B}_{-2,-1} := 2(c-1)(c-2)-(a-b)(c-b-1),$$

$$\begin{aligned}\mathcal{B}_{-2,-2} := {} & 2(c-1)(c-2)\{(2c-1)(a+b-1)-a(a+1)-b(b+1)+2\} \\ & -(a-b-1)(a-b+1)\{(c-1)(2c-a-b-3)+ab\},\end{aligned}$$

$$\begin{aligned}\mathcal{B}_{2,2} := {} & 2c(c+1)\{(2c+1)(a+b-1)-a(a-1)-b(b-1)\} \\ & -(a-b-1)(a-b+1)\{(c+1)(2c-a-b+1)+ab\},\end{aligned}$$

$$\mathcal{B}_{-1,-2} := 2(c-1)(a+b-1)-(a-b)^2+1,$$

$$\mathcal{B}_{0,2} := a(2c-a)+b(2c-b)-2c+1,$$

$$\mathcal{B}_{0,-2} := a(2c-a)+b(2c-b)-2c+1,$$

$$\mathcal{B}_{1,2} := 2c(a+b-1)-(a-b)^2+1,$$

$$\mathcal{B}_{2,1} := 2c(c+1)-(a-b)(c-b+1),$$

$$\mathcal{B}_{2,0} := (c-a+1)(c-b+1)+c(c+1),$$

$$\mathcal{B}_{2,-2} := c(a+b-1)-(a-1)(b-1).$$

Table 3. Table for $C_{j,i}$

$j\backslash i$	2	1	0	−1	−2
−2	−4	$-(c-a-1)$	4	$\mathcal{C}_{-2,-1}$	$\mathcal{C}_{-2,-2}$
−1	$-(4c-a-b-3)$	−1	1	$2c+a-b-2$	$\mathcal{C}_{-1,-2}$
0	−8	−1	0	1	8
1	$\mathcal{C}_{1,2}$	$-(2c+a-b)$	−1	1	$4c-a-b+1$
2	$\mathcal{C}_{2,2}$	$\mathcal{C}_{2,1}$	−4	$c-a+1$	4

Here

$$\mathcal{C}_{-2,-1} := 2(c-1)(c-2)+(a-b)(c-a-1),$$

$$\mathcal{C}_{-2,-2} := 4(2c-a+b-3)(2c+a-b-3),$$

$$\mathcal{C}_{-1,-2} := 8c^2-2(c-1)(a+b+7)-(a-b)^2-7,$$

$$\mathcal{C}_{1,2} := -8c^2+2c(a+b-1)+(a-b)^2-1,$$

$$\mathcal{C}_{2,2} := -4(2c+a-b+1)(2c-a+b+1),$$

$$\mathcal{C}_{2,1} := -2c(c+1)-(a-b)(c-a+1).$$

Table 4. Table for $\mathcal{D}_{i,j}$

$i\backslash j$	−2	−1	0	1	2
2	$\mathcal{D}_{2,-2}$	$\mathcal{D}_{2,-1}$	$\mathcal{D}_{2,0}$	$\mathcal{D}_{2,1}$	$\mathcal{D}_{2,2}$
1	$\frac{a(c+2n-1)}{(c-1)(a+4n)}$	$\frac{a}{a+4n}$	$\frac{a}{a+4n}$	$\frac{a(2c-a-4n)}{(2c-a)(a+4n)}$	$\mathcal{D}_{1,2}$
0	$\mathcal{D}_{0,-2}$	1	1	1	$\mathcal{D}_{0,2}$
−1	$\mathcal{D}_{-1,-2}$	$\frac{2c-a-4n-2}{2c-a-2}$	1	1	$\frac{c+2n+1}{c+1}$
−2	$\mathcal{D}_{-2,-2}$	$\mathcal{D}_{-2,-1}$	$\mathcal{D}_{-2,0}$	1	$\mathcal{D}_{-2,2}$

Here

$$\mathcal{D}_{2,-2} := \frac{(a+1)\{(c-1)(a-1)+2n(a+2n)\}}{(c-1)(a+4n-1)(a+4n+1)},$$

$$\mathcal{D}_{2,-1} := \frac{(a+1)(a-1)}{(a+4n+1)(a+4n-1)},$$

$$\mathcal{D}_{2,0} := \frac{(a+1)\{(a-1)(2c-a-1)-4n(a+2n)\}}{(2c-a-1)(a+4n+1)(a+4n-1)},$$

$$\mathcal{D}_{2,1} := \frac{(a+1)\{(a-1)(2c-a-1)-8n(a+2n)\}}{(2c-a-1)(a+4n+1)(a+4n+1)},$$

$$\mathcal{D}_{2,2} := \frac{N_{2,2}}{(c+1)(2c-a+1)(2c-a-1)(a+4n+1)(a+4n-1)},$$

where

$$\begin{aligned} N_{2,2} := (a+1)\{&(a-1)(c+1)(2c-a+1)(2c-a-1) \\ &- 2an(6c+a+5)(2c-a+1) \\ &+ 4n^2(5a^2+4a-5-4c(3c-a+4)+64n^3(a+n)\}, \end{aligned}$$

$$\mathcal{D}_{1,2} := \frac{a\{(c+1)(2c-a)-2n(2c+a+4n+2)\}}{(c+1)(2c-a)(a+4n)},$$

$$\mathcal{D}_{0,-2} := 1-\frac{2a(a+2n)}{(c-1)(2c-a-3)},$$

$$\mathcal{D}_{0,2} := 1-\frac{2n(a+2n)}{(c+1)(2c-a+1)},$$

$$\mathcal{D}_{-1,-2} := 1-\frac{2n(2c+a+4n-2)}{(c-1)(2c-a-4)},$$

$$\mathcal{D}_{-2,-2} := 1-\frac{N_{-2,-2}}{(a-1)(c-1)(2c-a-3)(2c-a-5)},$$

where

$$N_{-2,-2} := 2an(6c + a - 7)(2c - a - 3)$$

$$- 4n^2\{5a^2 - 4a - 21 - 4c(3c - a - 8)\} - 64n^3(a + n),$$

$$\mathcal{D}_{-2,-1} := 1 - \frac{8n(a + 2n)}{(a - 1)(2c - a - 3)},$$

$$\mathcal{D}_{-2,0} := 1 - \frac{4n(a + 2n)}{(a - 1)(2c - a - 1)},$$

$$\mathcal{D}_{-2,2} := 1 + \frac{2n(a + 2n)}{(c + 1)(a - 1)}.$$

Table 5. Table for $\mathcal{E}_{i,j}$

$i \backslash j$	-2	-1	0	1	2
2	$\mathcal{E}_{2,-2}$	$\mathcal{E}_{2,-1}$	$\mathcal{E}_{2,0}$	$\mathcal{E}_{2,1}$	$\mathcal{E}_{2,2}$
1	$\frac{c-a-2n-2}{(c-1)(a+4n+2)}$	$\frac{2c-a-2}{(2c-1)(a+4n+2)}$	$\frac{1}{a+4n+2}$	$\frac{2c+a+4n+2}{(2c+1)(a+4n+2)}$	$\mathcal{E}_{1,2}$
0	$\frac{1}{1-c}$	$\frac{1}{1-2c}$	0	$\frac{1}{1+2c}$	$\frac{1}{1+c}$
-1	$\mathcal{E}_{-1,-2}$	$\frac{2c+a+4n}{a(1-2c)}$	$-\frac{1}{a}$	$\frac{a-2c}{a(2c+1)}$	$\frac{a-c+2n}{a(c+1)}$
-2	$\mathcal{E}_{-2,-2}$	$\mathcal{E}_{-2,-1}$	$\frac{2}{1-a}$	$\frac{4c-a+1}{(1-a)(2c+1)}$	$\frac{2c-a+1}{(1-a)(c+1)}$

Here

$$\mathcal{E}_{2,-2} := \frac{(a + 1)(2c - a - 3)}{(c - 1)(a + 4n + 1)(a + 4n + 3)},$$

$$\mathcal{E}_{2,-1} := \frac{(a + 1)(4c - a - 3)}{(2c - 1)(a + 4n + 1)(a + 4n + 3)},$$

$$\mathcal{E}_{2,0} := \frac{2(a + 1)}{(a + 4n + 1)(a + 4n + 3)},$$

$$\mathcal{E}_{2,1} := \frac{(a+1)\{(4c+a+3)(2c-a-1)-8n(a+2n+2)\}}{(a+4n+1)(a+4n+3)(2c+1)(2c-a-1)},$$

$$\mathcal{E}_{2,2} := \frac{(a+1)(2c+a+4n+3)(2c-a-4n-1)}{(a+4n+1)(a+4n+3)(c+1)(2c-a-1)},$$

$$\mathcal{E}_{1,2} := \frac{(c+a+2)(2c-1)-2n(3a-2c+4n+2)}{(c+1)(2c-a)(a+4n+2)},$$

$$\mathcal{E}_{-1,-2} := \frac{(c+a)(2c-a-4)-2n(3a-2c+4n+6)}{a(a-2c+4)c-1},$$

$$\mathcal{E}_{-2,-2} := \frac{(2c+a+4n-1)(2c-a-4n-5)}{(1-a)(c-1)(2c-a-5)},$$

$$\mathcal{E}_{-2,-1} := \frac{(4c+a-1)(2c-a-3)-8n(a+2n+2)}{(a-1)(a-2c+3)(2c-1)}.$$

Remark 2. For double integrals similar to those presented here, we refer to Choi and Rathie [2].

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