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ON SEVERAL NEW CONTIGUOUS FUNCTION RELATIONS FOR k-HYPERGEOMETRIC FUNCTION WITH TWO PARAMETERS

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Abstract. Very recently, Mubeen, et al. [6] have obtained fifteen contiguous function relations for k-hypergeometric functions with one parameter by the same technique developed by Gauss. The aim of this paper is to obtain seventy-two new and interesting contiguous function relations for k-hypergeometric functions with two parameters. Obviously, for $k \to 1$ we recover the results obtained by Cho, et al. [2] and Rakha, et al. [8].

1. Introduction

The hypergeometric function ${}_2F_1[a, b; c; z]$ plays an important role in mathematical analysis and its applications. Gauss first introduced and studied hypergeometric series, paying special attention to the cases when a series can be summed into an elementary function. This indicated the elementary function and several other important functions in mathematics that can be expressed in terms of hypergeometric functions.

On the other hand, in the theory of hypergeometric function, the terminology contiguous function was introduced for the case in which one of the parameters is shifted by ± 1. For example, ${}_2F_1[a + 1, b; c; z]$ is contiguous to ${}_2F_1[a, b; c; z]$. Gauss defined two hypergeometric function to be contiguous if they have the same power series variable and if two of the parameters are pairwise equal and the third pair differs by ± 1. He proved that between ${}_2F_1[a, b; c; z]$ and any two of its contiguous functions, there exist a linear relation with coefficients almost linear in z. Since there are six contiguous functions to a given function ${}_2F_1$, Gauss [4] got fifteen relations. In fact, only four of the fifteen are independent as all others may be obtained by elimination, and use of the fact that ${}_2F_1$ is symmetric in a and b.

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With the help of these fifteen contiguous function relations Cho, et al. [2] obtained twenty four contiguous function relations for two parameters. Since in all seventy two such type of relation exist, therefore in 2000, Rakha, et al. [8] obtained remaining forty eight results.

Very recently, Mubeen, et al. [6] obtained fifteen contiguous function relations for k-hypergeometric functions with one parameter by the same technique developed by Gauss.

The aim of this paper is to obtain seventy two new and interesting contiguous function relations for k-hypergeometric functions with two parameters. Obviously, for $k \to 1$ we recover the results obtained by Cho, et al. [2] and Rakha, et al. [8].

For this, we wish to include a brief history of contiguous function relations for k-hypergeometric functions and other related topics here, so that this paper may be self-contained. The same will be given in the next section.

Diaz et al. [3] introduced k-gamma and k-beta function ($k > 0$) and derived a number of properties. In 2009, Mansour [5] obtained the k-generalized gamma function $\Gamma_k(x)$ by functional equation. In 2012, Mubeen and Habibulla [6] also gave a useful and simple integral representation of some confluent k-hypergeometric functions for ${}_1F_{1,k}$ and ${}_2F_{1,k}$.

2. Preliminaries and results used

• **Pochhammer k-symbol**: Let $k > 0$. Then the Pochhammer k-symbol is defined by

$$(a)_{n,k} = a(a+1k)(a+2k)\cdots(a+(n-1)k) \text{ for } n \geq 1, a \neq 0, \text{ and } (a)_{0,k} = 0.$$

• **k-gamma function**: For $k > 0$ and $z \in \mathbb{C}$, the k-gamma function Γ_k is defined as

$$\Gamma_k(z) = \lim_{n\to\infty} \frac{n!k^n\,(nk)^{\frac{z}{k}-1}}{(z)_{n,k}}.$$

Its integral representation is given by

$$\Gamma_k(z) = \int_0^\infty t^{z-1} e^{\frac{-t^k}{k}}\,dt.$$

The relation between Pochhammer k-symbol and k-gamma function is given as

$$(z)_{n,k} = \frac{\Gamma_k(z+nk)}{\Gamma_k(z)}.$$

Furthermore, we can write k-gamma function in terms of ordinary gamma function in the following form

$$\Gamma_k\,(z) = k^{\frac{z}{k}-1}\Gamma_k\left(\frac{z}{k}\right).$$

• ***k*-hypergeometric function**: The k-hypergeometric function with three parameters a, b and c is defined as

$$_2F_{1,k}(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_{n,k}\,(b)_{n,k}\,z^n}{(c)_{n,k}\,n!}$$

for $|z| < 1$ and for all $a, b, c \in \mathbb{C}$, $k > 0$, and $c \neq 0, -1, -2, \ldots$.

3. Contiguous function relation for *k*-hypergeometric function with one parameter

If we increase or decrease one and only one of the parameters of k-hypergeometric function,

$$_2F_{1,k}(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_{n,k}\,(b)_{n,k}\,z^n}{(c)_{n,k}\,n!},$$

by $\pm k$, then the resultant function is said to be contiguous to $_2F_{1,k}$.

For simplicity, we use the following notations

$$F_k = F_k(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_{n,k}\,(b)_{n,k}\,z^n}{(c)_{n,k}\,n!}$$

$$F_k(a+) = F_k(a+k,b;c;z) = \sum_{n=0}^{\infty} \frac{(a+k)_{n,k}\,(b)_{n,k}\,z^n}{(c)_{n,k}\,n!}, \text{ and}$$

$$F_k(a-) = F_k(a-k,b;c;z) = \sum_{n=0}^{\infty} \frac{(a-k)_{n,k}\,(b)_{n,k}\,z^n}{(c)_{n,k}\,n!}.$$

Similarly, we can write the notations for $F_k(b+)$, $F_k(c+)$, $F_k(b-)$ and $F_k(c-)$.

Let $\delta_{n,k} = \frac{(a)_{n,k}(b)_{n,k}z^n}{(c)_{n,k}n!}$. Then

$$F_k = F_k(a,b;c;z) = \sum_{n=0}^{\infty} \delta_{n,k}.$$

Now consider

$$F_k(a+) = \sum_{n=0}^{\infty} \frac{(a+k)_{n,k}\,(b)_{n,k}\,z^n}{(c)_{n,k}\,n!}.$$

Since $a(a+k)_{n,k} = (a)_{n,k}(a+nk)$, therefore, by using this result, we get

$$F_k(a+) = \sum_{n=0}^{\infty} \frac{(a+nk)}{a}\delta_{n,k}.$$

Similarly, we can write $F_k(a-)$ as

$$F_k(a-) = \sum_{n=0}^{\infty} \frac{(a-k)}{(a+(n-1)k)}\delta_{n,k},$$

by using the result $(a+(n-1)k)(a-k)_{n,k} = (a-k)(a)_{n,k}$.

Similarly, for $F_k(b+)$, $F_k(c+)$, $F_k(b-)$ and $F_k(c-)$.

Using the differential operator $k\theta = kz\left(\frac{d}{dz}\right)$, Mubeen [6], in 2000 has obtained the following 15 k-contiguous functions relations by the same technique given in Rainville book [7].

3.1. Fifteen contiguous function relations for k-hypergeometric function established by Mubeen

$$(a-b)\,F_k = aF_k\,(a+) - bF_k\,(b+)\,, \tag{3.1}$$

$$(a-c+k)\,F_k = aF_k\,(a+) - (c-k)\,F_k\,(c-)\,, \tag{3.2}$$

$$(a+(b-c)\,kz)\,F_k = a\,(1-kz)\,F_k\,(a+) - c^{-1}\,(c-a)\,(c-b)\,kzF_k\,(c+)\,, \tag{3.3}$$

$$(1-kz)\,F_k = F_k\,(a-) - c^{-1}\,(c-b)\,kzF_k\,(c+)\,, \tag{3.4}$$

$$(1-kz)\,F_k = F_k\,(b-) - c^{-1}\,(c-a)\,kzF_k\,(c+)\,, \tag{3.5}$$

$$(2a-c+(b-a)\,kz)\,F_k = a\,(1-kz)\,F_k\,(a+) - (c-a)\,F_k\,(a-)\,, \tag{3.6}$$

$$(a+b-c)\,F_k = a\,(1-kz)\,F_k\,(a+) - (c-b)\,F_k\,(b-)\,, \tag{3.7}$$

$$(a+b-c)\,F_k = (a-c)\,F_k\,(a-) + b\,(1-kz)\,F_k\,(b+)\,, \tag{3.8}$$

$$(b-a)\,(1-k\;z)\,F_k = (c-a)\,F_k\,(a-) - (c-b)\,F_k\,(b-)\,, \tag{3.9}$$

$$(k-a+(c-b-k)\,kz)\,F_k = (c-a)\,F_k\,(a-) - (c-k)\,(1-kz)\,F_k\,(c-)\,, \tag{3.10}$$

$$(2b-c+(a-b)\,kz)\,F_k = b\,(1-kz)\,F_k\,(b+) - (c-b)\,F_k\,(b-)\,, \tag{3.11}$$

$$(b+(a-c)\,kz)\,F_k = b\,(1-kz)\,F_k\,(b+) - c^{-1}\,(c-a)\,(c-b)\,kzF_k\,(c+)\,, \tag{3.12}$$

$$(b-c+k)\,F_k = bF_k\,(b+) - (c-k)\,F_k\,(c-)\,, \tag{3.13}$$

$$(k-b+(c-a-k)\,kz)\,F_k = (c-b)\,F_k\,(b-) - (c-k)\,(1-kz)\,F_k\,(c-)\,, \tag{3.14}$$

$$(c-k+(a+b+k-2c)\,kz)\,F_k \tag{3.15}$$

$$= (c-k)(1-kz)F_k(c-) - c^{-1}(c-a)(c-b)kzF_k(c+).$$

Remark. It is easy to see that in (3.1) to (3.15) if we take $k \to 1$ we get Gauss's fifteen contiguous function relations.

4. Contiguous function relations for k-hypergeometric function with two parameters

We now define,

$$F_k(a+,b+) = F_k(a+k, b+k; c; z) = \sum_{n=0}^{\infty} \frac{(a+k)_{n,k}(b+k)_{n,k} z^n}{(c)_{n,k} n!}$$

and similar expression for $F_k(a+,b-), F_k(a+,c+), F_k(a+,c-), \ldots$. With this definition, in this section the following twenty four new and interesting results will be obtained with the help of the results (3.1) to (3.5).

4.1. Main results

(4.1) $(a-k)F_k = (a-b-k)F_k(a-) + bF_k(a-,b+),$

(4.2) $(b-k)F_k = (b-a-k)F_k(b-) + aF_k(a+,b-),$

(4.3) $(a-k)F_k = (a-c)F_k(a-) + (c-k)F_k(a-,c-),$

(4.4) $cF_k = (c-a)F_k(c+) + aF_k(a+,c+),$

(4.5) $(a-k)(1-kz)F_k$
$= [a-k+(b-c)kz]F_k(a-) + c(c-a+k)(c-b)kzF_k(a-,c+),$

(4.6) $(c-k)^{-1}(c-a-k)(c-b-k)kzF_k$
$= [-a-(b-c+k)kz]F_k(c-) + a(1-kz)F_k(a+,c-),$

(4.7) $F_k = (1-kz)F_k(a+) + c^{-1}(c-b)kzF_k(a+,c+),$

(4.8) $(c-k)^{-1}(c-b-k)kzF_k = (kz-1)F_k(c-) + F_k(a-,c-),$

(4.9) $F_k = (1-kz)F_k(b+) + c^{-1}(c-a)kzF_k(b+,c+),$

(4.10) $(c-k)^{-1}(c-a-k)kzF_k = (kz-1)F_k(c-) + F_k(b-,c-),$

(4.11) $(a-k)(1-kz)F_k = (a+b-c-k)F_k(a-) + (c-b)F_k(a-,b-),$

(4.12) $(c-b-k)F_k = (c-a-b-k)F_k(b+) + a(1-kz)F_k(a+,b+),$

(4.13) $(c-a-k)F_k = (c-a-b-k)F_k(a+) + b(1-kz)F_k(a+,b+),$

(4.14) $(b-k)(1-kz)F_k = (a+b-c-k)F_k(b-) + (c-a)F_k(a-,b-),$

(4.15) $(c-a-k)F_k = (b-a-k)(1-kz)F_k(a+) + (c-b)F_k(a+,b-),$

(4.16) $(c-b-k)F_k = (a-b-k)(1-kz)F_k(b+) + (c-a)F_k(a-,b+),$

(4.17) $(c-a-k)F_k$
$= [-a+(c-b-k)kz]F_k(a+) + (c-k)(1-kz)F_k(a+,c-),$

(4.18) $c(1-kz)F_k = (a-k-(c-b)kz)F_k(c+) + (c-a+k)F_k(a-,c+),$

(4.19) $(b-k)(1-kz)F_k$
$$= (b-k+(a-c)kz)F_k(b-) + c^{-1}(c-a)(c-b+k)kzF_k(b-,c+),$$

(4.20) $(c-k)^{-1}(c-a-k)(c-b-k)kzF_k$
$$= (-b-(a-c+k)kz)F_k(c-) + b(1-kz)F_k(b+,c-),$$

(4.21) $(b-k)F_k = (b-c)F_k(b-) + (c-k)F_k(b-,c-),$

(4.22) $cF_k = (c-b)F_k(c+) + bF_k(b+,c+),$

(4.23) $(c-b-k)F_k$
$$= (-b+(c-a-k)kz)F_k(b+) + (c-k)(1-kz)F_k(b+,c-),$$

(4.24) $c(1-kz)F_k = (b-k-(c-a)kz)F_k(c+) + (c-b+k)F_k(b-,c+).$

4.2. Derivations

The derivation of our new and interesting contiguous functions relations (4.1) to (4.24) are quite straightforward. For example, in the result (3.1) if we change a to $(a-k)$ we immediately get the result (4.1).

In exactly the same manner, other results also can be obtained. The scheme is outlined in Table 1 including (4.1).

Remark. The results

$$F_k(a-) - F_k(b-) + c^{-1}(b-a)kzF_k(c+) = 0,$$
$$F_k = F_k(a-,b+) + c^{-1}(b+k-a)kzF_k(b+,c+),$$
$$(c-k-b)F_k = (c-a)F_k(a-,b+) + (a-k-b)(1-kz)F_k(b+)$$

are obtained by Mubeen [6].

Table 1: 24 New Contiguous Functions Relations

Mubeen's	Change	Our New Results
(3.1)	$a \to a-$	(4.1)
(3.1) $(a \leftrightarrow b)$	$b \to b-$	(4.2)
(3.2)	$a \to a-$	(4.3)
(3.2)	$c \to c+$	(4.4)
(3.3)	$a \to a-$	(4.5)
(3.3)	$c \to c-$	(4.6)
(3.4)	$a \to a+$	(4.7)
(3.4)	$c \to c-$	(4.8)
(3.5)	$b \to b+$	(4.9)
(3.4) $(a \leftrightarrow b)$	$c \to c-$	(4.10)

(3.8) $(a \leftrightarrow b)$	$a \to a-$	(4.11)
(3.8) $(a \leftrightarrow b)$	$b \to b+$	(4.12)
(3.8)	$a \to a+$	(4.13)
(3.8)	$b \to b-$	(4.14)
(3.9)	$a \to a+$	(4.15)
(3.9) $(a \leftrightarrow b)$	$b \to b+$	(4.16)
(3.10)	$a \to a+$	(4.17)
(3.10)	$c \to c+$	(4.18)
(3.12)	$b \to b-$	(4.19)
(3.3) $(a \leftrightarrow b)$	$c \to c-$	(4.20)
(3.13)	$b \to b-$	(4.21)
(3.13)	$c \to c+$	(4.22)
(3.10) $(a \leftrightarrow b)$	$b \to b+$	(4.23)
(3.10) $(a \leftrightarrow b)$	$c \to c+$	(4.24)

We conclude this section by remarking that rest forty eight relations will be obtained in the next section with the help of fifteen Mubeen's contiguous function relations [6] and twenty four new contiguous function relationswhich are given in the above section.

5. Further forty-eight contiguous functions relations for k-hypergeometric functions with two parameters

In this section, we shall establish the following forty-eight new and interesting contiguous function relations for k-hypergeometric function with two parameters.

5.1. Main results

(5.1) $$[(a-b)(a-b-k)(1-kz)+b(c-b-k)]F_k$$
$$= a(a-b-k)(1-kz)F_k(a+)+b(c-a)F_k(a-,b+),$$

(5.2) $$[a(a+2b-2c-k)-c(b-c)-(k-b)(b-c)kz]F_k$$
$$= a(a+b-c-k)(1-kz)F_k(a+)+(c-b)(c-a)F_k(a-,b-),$$

(5.3) $$[a-(a-b)kz]F_k = a(1-kz)F_k(a+)+c^{-1}b(c-a)kzF_k(b+,c+),$$

(5.4) $$(c-a-k)[b+(a-b)kz]F_k$$
$$= a[(c-a-k)kz-b]F_k(a+)+b(c-k)(1-kz)F_k(b+,c-),$$

(5.5) $[a(b-k)+\{(c-a)(c-a-b)+(b-k)(b-c)\}kz]F_k$
$= a(1-kz)[(b-k)-(c-a)kz]F_k(a+)$
$+ c^{-1}(c-a)(c-b)(c-b-k)kzF_k(b-,c+),$

(5.6) $(a-c+k)F_k = a(1-kz)F_k(a+)-(c-k)F_k(b-,c-),$

(5.7) $[a(a-k)+(c-b)(c-3a+k)kz+(b-a)(b-c)k^2z^2]F_k$
$= a(1-kz)[(a-k)-(c-b)kz]F_k(a+)$
$+ c^{-1}(c-a)(c-b)(c-b+k)kzF_k(a-,c+),$

(5.8) $[(c-a-k)+(a-b)kz]F_k$
$= a(kz-1)F_k(a+)+(c-k)F_k(a-,c-),$

(5.9) $[(c-a)(c-a-b-k)+ab-b(c-b-k)kz]F_k$
$= (c-a)(c-a-b-k)F_k(a-)+ab(1-kz)^2F_k(a+,b+),$

(5.10) $[c(b-k)+a(a-2b+k)-(b-a-k)(b-a)kz]F_k$
$= (c-a)(b-a-k)F_k(a-)+a(c-b)F_k(a+,b-),$

(5.11) $F_k = F_k(a-)+c^{-1}bkzF_k(b+,c+),$

(5.12) $[b(a-k)+kz\{(c-a-b)(c-a-k)-b(c-b-k)\}]F_k$
$= (c-a)[(c-a-k)kz-b]F_k(a-)+b(c-k)(1-kz)^2F_k(b+,c-),$

(5.13) $[(b-k)(1-kz)+(a-b)kz(1-kz)]F_k$
$= [(b-k)+(a-c)kz]F_k(a-)+c^{-1}(c-b)(c-b+k)kzF_k(b-,c+),$

(5.14) $[(a-k)+(b-a)kz]F_k = (a-c)F_k(a-)+(c-k)F_k(b-,c-),$

(5.15) $[(c-a)-(b-a)kz]F_k$
$= (c-a)F_k(a-)+ac^{-1}(c-b)kzF_k(a+,c+),$

(5.16) $[a(a-k)+(c-3a)(c-b-k)kz+(a-b)(c-b-k)k^2z^2]F_k$
$= (c-a)[(c-b-k)kz-a]F_k(a-)+a(c-k)(1-kz)^2F_k(a+,c-),$

(5.17) $[(b-a)(b-a-k)(1-kz)+a(c-a-k)]F_k$
$= b(b-a-k)(1-kz)F_k(b+)+a(c-b)F_k(a+,b-),$

(5.18) $[b(b+2a-2c-k)-c(a-c)-(k-a)(a-c)kz]F_k$
$= b(a+b-c-k)(1-kz)F_k(b+)+(c-a)(c-b)F_k(a-,b-),$

(5.19) $[b(b-k)+(c-a)(c-3b+k)kz+(a-b)(a-c)k^2z^2]F_k$
$= b[(b-k)-(c-a)kz](1-kz)F_k(b+)$
$+ c^{-1}(c-b)(c-a)(c-b+k)kzF_k(b-,c+),$

(5.20) $[(c-b-k)+(b-a)kz]F_k$
$= b(kz-1)F_k(b+)+(c-k)F_k(b-,c-),$

(5.21) $[b-(b-a)kz]F_k$

$$= b(1-kz)F_k(b+) + c^{-1}a(c-b)kzF_k(a+,c+),$$

(5.22) $$[b(a-k) + \{(c-b)(c-a-b) + (a-k)(a-c)\}kz]F_k$$
$$= b(1-kz)[(a-k)-(c-b)kz]F_k(b+)$$
$$+ c^{-1}(c-a)(c-b)(c-a+k)kzF_k(a+,c+),$$

(5.23) $$(c-b-k)[a+(b-a)kz]F_k$$
$$= b[(c-b-k)kz-a]F_k(b+) + a(c-k)(1-kz)F_k(a+,c-),$$

(5.24) $$(b-c+k)F_k = b(1-kz)F_k(b+) - (c-k)F_k(a-,c-),$$

(5.25) $$[(c-b)(c-a-b-k) + ab - a(c-a-k)kz]F_k$$
$$= (c-b)(c-a-b-k)F_k(b-) + ab(1-kz)^2F_k(a+,b+),$$

(5.26) $$[(a-k)c + b(k+b-2a) - (a-b-k)(a-b)kz]F_k$$
$$= (c-b)(a-b-k)F_k(b-) + b(c-a)F_k(a-,b-),$$

(5.27) $$[(c-b)-(a-b)kz]F_k$$
$$= (c-b)F_k(b-) + bc^{-1}(c-a)kzF_k(b+,c+),$$

(5.28) $$[b(b-k) + (c-3b)(c-a-k)kz + (b-a)(c-a-k)k^2z^2]F_k$$
$$= (c-b)[(c-a-k)kz-b]F_k(b-) + b(c-k)(1-kz)^2F_k(b+,c-),$$

(5.29) $$F_k = F_k(b-) + c^{-1}akzF_k(a+,c+),$$

(5.30) $$[(a-k)(1-kz) + (b-a)kz(1-kz)]F_k$$
$$= [(a-k)+(b-c)kz]F_k + (b-)(c-a)(c-a+k)kzF_k(a-,c+),$$

(5.31) $$[a(b-k) + z\{(c-a-b)(c-b-k) - a(c-a-k)\}]F_k$$
$$= (c-b)[(c-b-k)kz-a]F_k(b-) + a(c-k)(1-kz)^2F_k(a+,c-),$$

(5.32) $$[(b-k)+(a-b)kz]F_k = (b-c)F_k(b-) + (c-k)F_k(a-,c-),$$

(5.33) $$[ab - kz\{(c-b-k)(a+b-c) - a(a-c)\}]F_k$$
$$= c^{-1}(c-a)(c-b)(c-a-b-k)kzF_k(c+) + ab(1-kz)^2F_k(a+,b+),$$

(5.34) $$[a+(b-a-k)kz]F_k$$
$$= aF_k(a+,b-) + c^{-1}(c-a)(b-a-k)kzF_k(c+),$$

(5.35) $$[b+(a-b-k)kz]F_k$$
$$= bF_k(a-,b+) + c^{-1}(c-b)(a-b-k)kzF_k(c+),$$

(5.36) $$(1-kz)F_k = c^{-1}(a+b-c-k)kzF_k(c+) + F_k(a-,b-),$$

(5.37) $$[b(c-k) + bkz(2a+b-3c+2k) - (a-c)(c-a-k)k^2z^2]F_k$$
$$= c^{-1}(c-a)(c-b)kz[(c-a-k)kz-b]F_k(c+)$$
$$+ b(c-k)(1-k)^2F_k(b+,c-),$$

(5.38) $$[(c-k)+(b-c)kz]F_k$$

$$= c^{-1}(c-a)(b-c)kzF_k(c+) + (c-k)F_k(b-, c-),$$

(5.39) $$[a(c-k) + akz(a+2b-3c+2k) - (b-c)(c-b-k)k^2z^2]F_k$$
$$= c^{-1}(c-a)(c-b)kz[(c-b-k)kz - a]F_k(c+)$$
$$+ a(c-k)(1-kz)^2F_k(a+, c-),$$

(5.40) $$[(c-k) + (a-c)kz]F_k$$
$$= c^{-1}(c-b)(a-c)kzF_k(c+) + (c-k)F_k(a-, c-),$$

(5.41) $$(c-b-k)(c-a-k)F_k$$
$$= (c-k)(c-a-b-k)F_k(c-) + ab(1-kz)F_k(a+, b+),$$

(5.42) $$(c-a-k)\{(b-k) - (b-a-k)kz\}F_k$$
$$= (c-k)(b-a-k)(1-kz)F_k(c-) + a(c-b)F_k(a+, b-),$$

(5.43) $$(c-b-k)\{(a-k) - (a-b-k)kz\}F_k$$
$$= (c-k)(a-b-k)(1-kz)F_k(c-) + b(c-a)F_k(a-, b+),$$

(5.44) $$[(a-k)(b-k) + (a^2+b^2+c^2+ab-2bc-2ac+ck-k^2)kz]F_k$$
$$= (c-k)(a+b-c-k)(1-kz)F_k(c-) + (c-a)(c-b)F_k(a-, b-),$$

(5.45) $$[(c-k) + (b-c+k)kz]F_k$$
$$= (c-k)(1-kz)F_k(c-) + c^{-1}b(c-a)kzF_k(b+, c+),$$

(5.46) $$[(c-k)(b-k) + (b-k)(2a+b-3c+k)kz$$
$$- (c-a)(k+a-c)k^2z^2]F_k$$
$$= (c-k)[(b-k) - (c-a)kz]$$
$$+ c^{-1}(c-a)(c-b)(c-b+k)kzF_k(b-, c+),$$

(5.47) $$[(c-k) + (a-c+k)kz]F_k$$
$$= (c-k)(1-kz)F_k(c-) + c^{-1}a(c-b)kzF_k(a+, c+),$$

(5.48) $$[(c-k)(a-k) + (a-k)(2b+a-3c+k)kz$$
$$- (c-b)(k+b-c)k^2z^2]F_k$$
$$= (c-k)[(a-k) - (c-b)kz](1-kz)F_k(c-)$$
$$+ c^{-1}(c-b)(c-a)(c-a+k)kzF_k(a-, c+).$$

5.2. Derivations

The derivations of the above contiguous function relations (5.1) to (5.48) are quite straight forward. By algebraic manipulation, for example, if we wish to derive the result (5.1), then by use of the twenty-four new relations we obtained in Section 4, we can use (4.1) to eliminate $F_k(a-)$. In a similar manner, other results can be easily obtained, The scheme is outlined in the following Table 2.

Table 2: 48 New Contiguous Functions Relations

Mubeen Result	New Result	Eliminate	Our New Result
(3.6) or (3.1)	(4.1) or (4.16)	$F_k\,(a-)$ or $F_k\,(b+)$	(5.1)
(3.6) or (3.7)	(4.11) or (4.14)	$F_k\,(a-)$ or $F_k\,(b-)$	(5.2)
(3.1) or (3.3)	(4.9) or (4.22)	$F_k\,(b+)$ or $F_k\,(c+)$	(5.3)
(3.1) or (3.2)	(4.23) or (4.20)	$F_k\,(b+)$ or $F_k\,(c-)$	(5.4)
(3.3) or (3.7)	(4.24) or (4.19)	$F_k\,(c+)$ or $F_k\,(b-)$	(5.5)
(3.7) or (3.2)	(4.21) or (4.10)	$F_k\,(b-)$ or $F_k\,(c-)$	(5.6)
(3.6) or (3.3)	(4.5) or 4. 18)	$F_k\,(a-)$ or $F_k\,(c+)$	(5.7)
(3.6) or (3.2)	(4.3) or 4. 8)	$F_k\,(a-)$ or $F_k\,(c-)$	(5.8)
(3.6) or (3.8)	(4.13) or 4. 12)	$F_k\,(a+)$ or $F_k\,(b+)$	(5.9)
(3.6) or (3.9)	(4.15) or (4.2)	$F_k\,(a+)$ or $F_k\,(b-)$	(5.10)
(3.8) or (3.4)	(4.9) or (4.22)	$F_k\,(b+)$ or $F_k\,(c+)$	(5.11)
(3.8) or (3.10)	(4.23) or (4.20)	$F_k\,(b+)$ or $F_k\,(c-)$	(5.12)
(3.4) or (3.9)	(4.24) or (4.19)	$F_k\,(c+)$ or $F_k\,(b-)$	(5.13)

(3.9) or (3.10)	(4.21) or (4.10)	$F_k\,(b-)$ or $F_k\,(c-)$	(5.14)
(3.6) or (3.4)	(4.7) or (4.4)	$F_k\,(a+)$ or $F_k\,(c+)$	(5.15)
(3.6) or (3.10)	(4.17) or (4.6)	$F_k\,(a+)$ or $F_k\,(c-)$	(5.16)
(3.1) or (3.11)	(4.15) or (4.2)	$F_k\,(a+)$ or $F_k\,(b-)$	(5.17)
(3.8) or) (3.11)	(4.11) or (4.14)	$F_k\,(a-)$ or $F_k\,(b-)$	(5.18)
(3.11) or (3.12)	(4.19) or (4.24)	$F_k\,(b-)$ or $F_k\,(c+)$	(5.19)
(3.11) or (3.13)	(4.21) or (4.10)	$F_k\,(b-)$ or $F_k\,(c-)$	(5.20)
(3.1) or (3.12)	(4.7) or (4.4)	$F_k\,(a+)$ or $F_k\,(c+)$	(5.21)
(3.8) or (3.12)	(4.5) or (4.18)	$F_k\,(a-)$ or $F_k\,(c+)$	(5.22)
(3.1) or (3.13)	(4.17) or (4.6)	$F_k\,(a+)$ or $F_k\,(c-)$	(5.23)
(3.8) or (3.13)	(4.3) or (4.8)	$F_k\,(a-)$ or $F_k\,(c-)$	(5.24)
(3.7) or (3.11)	(4.13) or (4.12)	$F_k\,(a+)$ or $F_k\,(b+)$	(5.25)
(3.11) or (3.9)	(4.16) or (4.1)	$F_k\,(b+)$ or $F_k\,(a-)$	(5.26)
(3.11) or (3.5)	(4.9) or (4.22)	$F_k\,(b+)$ or $F_k\,(c+)$	(5.27)

(3.11) or (3.14)	(4.23) or (4.20)	$F_k\,(b+)$ or $F_k\,(c-)$	(5.28)
(3.7) or (3.5)	(4.7) or (4.4)	$F_k\,(a+)$ or $F_k\,(c+)$	(5.29)
(3.5) or (3.9)	(4.18) or (4.5)	$F_k\,(c+)$ or $F_k\,(a-)$	(5.30)
(3.7) or (3.14)	(4.17) or (4.6)	$F_k\,(a+)$ or $F_k\,(c-)$	(5.31)
(3.9) or (3.14)	(4.3) or (4.8)	$F_k\,(a-)$ or $F_k\,(c-)$	(5.32)
(3.3) or (3.12)	(4.13) or (4.12)	$F_k\,(a+)$ or $F_k\,(b+)$	(5.33)
(3.3) or (3.5)	(4.15) or (4.2)	$F_k\,(a+)$ or $F_k\,(b-)$	(5.34)
(3.12) or (3.4)	(4.16) or (4.1)	$F_k\,(b+)$ or $F_k\,(a-)$	(5.35)
(3.4) or (3.5)	(4.11) or (4.14)	$F_k\,(a-)$ or $F_k\,(b-)$	(5.36)
(3.12) or (3.15)	(4.23) or (4.20)	$F_k\,(b+)$ or $F_k\,(c-)$	(5.37)
(3.5) or (3.15)	4. 21) or (4.10)	$F_k\,(b-)$ or $F_k\,(c-)$	(5.38)
(3.3) or (3.15)	(4.17) or (4.6)	$F_k\,(a+)$ or $F_k\,(c-)$	(5.39)
(3.4) or (3.15)	(4.3) or (4.8)	$F_k\,(a-)$ or $F_k\,(c-)$	(5.40)
(3.2) or (3.13)	(4.13) or (4.12)	$F_k\,(a+)$ or $F_k\,(b+)$	(5.41)

(3.2) or (3.14)	(4.15) or (4.2)	$F_k\,(a+)$ or $F_k\,(b-)$	(5.42)
(3.13) or (3.10)	(4.16) or 4. 1)	$F_k\,(b+)$ or $F_k\,(a-)$	(5.43)
(3.10) or (3.14)	(4.11) or (4.14)	$F_k\,(a-)$ or $F_k\,(b-)$	(5.44)
(3.13) or (3.15)	(4.9) or (4.22)	$F_k\,(b+)$ or $F_k\,(c+)$	(5.45)
(3.15) or (3.14)	(4.24) or (4.19)	$F_k\,(c+)$ or $F_k\,(b-)$	(5.46)
(3.2) or (3.15)	(4.7) or (4.4)	$F_k\,(a+)$ or $F_k\,(c+)$	(5.47)
(3.15) or (3.10)	(4.18) or (4.5)	$F_k\,(c+)$ or $F_k\,(a-)$	(5.48)

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