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NEW SERIES IDENTITIES FOR $\frac{1}{\Pi}$

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Abstract. In the theory of hypergeometric and generalized hypergeometric series, classical summation theorems have been found interesting applications in obtaining various series identities for Π, Π^2 and $\frac{1}{\Pi}$. The aim of this research paper is to provide twelve general formulas for $\frac{1}{\Pi}$. On specializing the parameters, a large number of very interesting series identities for $\frac{1}{\Pi}$ not previously appeared in the literature have been obtained. Also, several other results for multiples of Π, Π^2, $\frac{1}{\Pi^2}$, $\frac{1}{\Pi^3}$ and $\frac{1}{\sqrt{\Pi}}$ have been obtained. The results are established with the help of the extensions of classical Gauss's summation theorem available in the literature.

1. Introduction

Roughly speaking, a generalized hypergeometric series to be a series $\sum C_n$ with term ratio $\frac{C_{n+1}}{C_n}$ a rational function of n. In general it can be defined as follows [2, 20, 23]

$$(1.1)\qquad {}_pF_q\left[\begin{array}{ccc} a_1, & \ldots, & a_p \\ & & & ; & z \\ b_1, & \ldots, & b_q \end{array}\right] = \sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!},$$

where $(a)_n$ is the well known Pochhammer's symbol (or the shifted or raised factorial) defined for every complex number a by

$$(1.2)\qquad (a)_n = \begin{cases} a(a+1)\cdots(a+n-1), \ n \in \mathbb{N} \\ 1, \ n = 0. \end{cases}$$

The gamma function is defined by the Euler integral

$$(1.3)\qquad \Gamma(x) = \int_0^{\infty} u^{x-1}e^{-u}du$$

provided $\mathrm{Re}(x) > 0$.

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In view of (1.3), (1.2) can be written as

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

For $p = 2$ and $q = 1$, (1.1) can be reduced to the well known hypergeometric series ${}_2F_1$. For convergence conditions of ${}_pF_q$ and ${}_2F_1$ and for other properties, we refer the standard texts [2, 20, 23].

In the theory of hypergeometric series ${}_2F_1$ and generalized hypergeometric series ${}_pF_q$, classical summation theorems such as those of Gauss, Gauss second, Bailey and Kummer for the series ${}_2F_1$; Watson, Dixon, Whipple and Saalschütz for the series ${}_3F_2$ and others play a key role.

Recently good progress has been done in generalizing and extending the above mentioned classical summation theorems. For this, we refer the research papers and books [11, 13, 14, 15, 19, 21, 24] and the references cited there in.

These classical summation theorems have been found applications in obtaining various series identities for Π, Π^2 and $\frac{1}{\Pi}$. For this, we refer, interesting papers and books [1, 4, 3, 5, 6, 7, 8, 9, 10, 12, 16, 17, 18, 22, 25, 26, 27, 28, 29] and the references cited there in.

In 2013, Liu, et al. [17] have obtained the following interesting general series identity

$$\begin{aligned}
&\frac{(2\gamma)_{2p}\left(\frac{1+\alpha}{2}\right)_m\left(\frac{1+\beta}{2}\right)_n\left(\gamma+\frac{1-\alpha}{2}\right)_{p-m}\left(\gamma+\frac{1-\beta}{2}\right)_{p-n}}{(\alpha)_{2m}(\beta)_{2n}(\gamma)_p\left(\frac{1}{2}+\gamma\right)_p\left(\gamma+\frac{1-\alpha-\beta}{2}\right)_{p-m-n}}\\
&\times\sum_{k=0}^{\infty}\frac{(\alpha)_{2m+k}(\beta)_{2n+k}(\gamma)_{p+k}}{k!\left(\frac{\alpha+\beta+1}{2}\right)_{m+n+k}(2\gamma)_{2p+k}}\\
&=\frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}+\gamma\right)\Gamma\left(\frac{\alpha+\beta+1}{2}\right)\Gamma\left(\gamma+\frac{1-\alpha-\beta}{2}\right)}{\Gamma\left(\frac{1+\alpha}{2}\right)\Gamma\left(\frac{1+\beta}{2}\right)\Gamma\left(\gamma+\frac{1-\alpha}{2}\right)\Gamma\left(\gamma+\frac{1-\beta}{2}\right)},
\end{aligned}\tag{1.4}$$

where $m, n, p \in \mathbb{N}$ with $p - m - n \geq 0$, by employing the following classical Watson's ${}_3F_2$-summation theorem [2]

$$\begin{aligned}
&{}_3F_2\left[\begin{matrix}\alpha, & \beta, & \gamma; & \\ & & & 1\\ \frac{1}{2}(\alpha+\beta+1) & 2\gamma & & \end{matrix}\right]\\
&=\frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}+\gamma\right)\Gamma\left(\frac{\alpha+\beta+1}{2}\right)\Gamma\left(\gamma+\frac{1-\alpha-\beta}{2}\right)}{\Gamma\left(\frac{1+\alpha}{2}\right)\Gamma\left(\frac{1+\beta}{2}\right)\Gamma\left(\gamma+\frac{1-\alpha}{2}\right)\Gamma\left(\gamma+\frac{1-\beta}{2}\right)}
\end{aligned}\tag{1.5}$$

provided $\mathrm{Re}(2\gamma - \alpha - \beta) > -1$.

As special cases, they have deduced the following series identities for $\frac{1}{\Pi}$.

$$\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k\left(\frac{3}{2}\right)_k}{k!\,(k+2)!}=\frac{8}{3\pi}, \tag{1.6}$$

$$\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k\left(\frac{3}{2}\right)_{k+1}}{k!\,(k+4)!}=\frac{32}{105\pi}, \tag{1.7}$$

$$\sum_{k=0}^{\infty}\frac{\left(\frac{1}{3}\right)_k\left(\frac{5}{3}\right)_k}{k!\,(k+2)!}=\frac{27\sqrt{3}}{20\pi}, \tag{1.8}$$

$$\sum_{k=0}^{\infty}\frac{\left(\frac{2}{3}\right)_k\left(\frac{4}{3}\right)_k}{k!\,(k+2)!}=\frac{27\sqrt{3}}{16\pi}, \tag{1.9}$$

$$\sum_{k=0}^{\infty}\frac{\left(\frac{1}{6}\right)_k\left(\frac{11}{6}\right)_k}{k!\,(k+2)!}=\frac{108}{55\pi}, \tag{1.10}$$

$$\sum_{k=0}^{\infty}\frac{\left(\frac{5}{6}\right)_k\left(\frac{7}{6}\right)_k}{k!\,(k+2)!}=\frac{108}{35\pi}, \tag{1.11}$$

$$\sum_{k=0}^{\infty}\frac{\left(\frac{1}{4}\right)_k\left(\frac{7}{4}\right)_k}{k!\,(k+2)!}=\frac{32\sqrt{2}}{21\pi}, \tag{1.12}$$

$$\sum_{k=0}^{\infty}\frac{\left(\frac{3}{4}\right)_k\left(\frac{5}{4}\right)_k}{k!\,(k+2)!}=\frac{32\sqrt{2}}{15\pi}, \tag{1.13}$$

$$\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k\left(\frac{1}{2}\right)_k}{k!\,(k+1)!}=\frac{4}{\pi}, \tag{1.14}$$

$$\sum_{k=0}^{\infty}\frac{\left(\frac{1}{3}\right)_k\left(\frac{2}{3}\right)_k}{k!\,(k+1)!}=\frac{9\sqrt{3}}{4\pi}, \tag{1.15}$$

$$\sum_{k=0}^{\infty}\frac{\left(\frac{3}{4}\right)_k\left(\frac{1}{4}\right)_k}{k!\,(k+1)!}=\frac{8\sqrt{2}}{3\pi}, \tag{1.16}$$

$$\sum_{k=0}^{\infty}\frac{\left(\frac{5}{6}\right)_k\left(\frac{1}{6}\right)_k}{k!\,(k+1)!}=\frac{18}{5\pi}. \tag{1.17}$$

Remark. The result (1.7) is a corrected form of the result given in [17].

Recently Mohammed et al. [18] pointed out that the above results (1.6) to (1.17) which where deduced from a general result (1.4) due to Liu, et al. [17] can be obtained very quickly by employing the following classical Gauss's summation theorem [20]

$$ {}_2F_1\left[\begin{matrix} a, & b; & \\ & & 1 \\ c & & \end{matrix}\right]=\frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}, \tag{1.18}$$

provided that $\mathrm{Re}(c-a-b)>0$.

A natural generalization of classical Gauss's summation theorem (1.18) is also available in the literature [19]

(1.19)

$$ {}_3F_2\left[\begin{matrix} a, & b, & d+1 \\ & c+1, & d \end{matrix} ; 1\right] = \frac{\Gamma(c+1)\,\Gamma(c-a-b)}{\Gamma(c-a+1)\,\Gamma(c-b+1)}\left[c-a-b+\frac{ab}{d}\right] $$

provided $\mathrm{Re}(c-a-b)>0$ and $d\neq 0,-1,-2,\ldots$.

The aim of this research paper is to provide the natural extensions of the results (1.6) to (1.17) by employing the extension (1.19) of classical Gauss's summation theorem. As special cases, we mention a large number of interesting series for $\frac{1}{\Pi}$.

The results established in this paper are simple interesting, easily established and may be potentially useful.

2. Main results

In this section, we shall establish twelve general series identities in the form of three theorems.

Theorem 2.1. *For $d\neq 0,-1,-2,\ldots$, the following results holds true*

$$ {}_3F_2\left[\begin{matrix} \frac{1}{2}, & \frac{5}{2}, & d+1 \\ & 6, & d \end{matrix} ; 1\right] = \frac{2048}{945\pi}\left[2+\frac{5}{4d}\right]. \tag{2.1} $$

Theorem 2.2. *For $d\neq 0,-1,-2,\ldots$, the following results holds true*

$$ {}_3F_2\left[\begin{matrix} \frac{1}{2}, & \frac{3}{2}, & d+1 \\ & 4, & d \end{matrix} ; 1\right] = \frac{64}{15\pi}\left[1+\frac{3}{4d}\right], \tag{2.2} $$

$$ {}_3F_2\left[\begin{matrix} \frac{1}{3}, & \frac{5}{3}, & d+1 \\ & 4, & d \end{matrix} ; 1\right] = \frac{729\sqrt{3}}{320\pi}\left[1+\frac{5}{9d}\right], \tag{2.3} $$

$$ {}_3F_2\left[\begin{matrix} \frac{2}{3}, & \frac{4}{3}, & d+1 \\ & 4, & d \end{matrix} ; 1\right] = \frac{729\sqrt{3}}{280\pi}\left[1+\frac{8}{9d}\right], \tag{2.4} $$

$$ {}_3F_2\left[\begin{matrix} \frac{1}{6}, & \frac{11}{6}, & d+1 \\ & 4, & d \end{matrix} ; 1\right] = \frac{23328}{6545\pi}\left[1+\frac{11}{36d}\right], \tag{2.5} $$

$$ {}_3F_2\left[\begin{matrix} \frac{5}{6}, & \frac{7}{6}, & d+1 \\ & 4, & d \end{matrix} ; 1\right] = \frac{23328}{5005\pi}\left[1+\frac{35}{36d}\right], \tag{2.6} $$

$$(2.7)\qquad {}_3F_2\left[\begin{matrix}\frac{1}{4}, & \frac{7}{4}, & d+1 \\ & 4, & d\end{matrix}\;;\;1\right] = \frac{1024\sqrt{2}}{385\pi}\left[1+\frac{7}{16d}\right],$$

$$(2.8)\qquad {}_3F_2\left[\begin{matrix}\frac{3}{4}, & \frac{5}{4}, & d+1 \\ & 4, & d\end{matrix}\;;\;1\right] = \frac{1024\sqrt{2}}{315\pi}\left[1+\frac{15}{16d}\right].$$

Theorem 2.3. *For $d \neq 0, -1, -2, \ldots$, the following results holds true*

$$(2.9)\qquad {}_3F_2\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2}, & d+1 \\ & 3, & d\end{matrix}\;;\;1\right] = \frac{32}{9\pi}\left[1+\frac{1}{4d}\right],$$

$$(2.10)\qquad {}_3F_2\left[\begin{matrix}\frac{1}{3}, & \frac{2}{3}, & d+1 \\ & 3, & d\end{matrix}\;;\;1\right] = \frac{81\sqrt{3}}{40\pi}\left[1+\frac{2}{9d}\right],$$

$$(2.11)\qquad {}_3F_2\left[\begin{matrix}\frac{3}{4}, & \frac{1}{4}, & d+1 \\ & 3, & d\end{matrix}\;;\;1\right] = \frac{256\sqrt{2}}{105\pi}\left[1+\frac{3}{16d}\right],$$

$$(2.12)\qquad {}_3F_2\left[\begin{matrix}\frac{1}{6}, & \frac{5}{6}, & d+1 \\ & 3, & d\end{matrix}\;;\;1\right] = \frac{1296}{385\pi}\left[1+\frac{5}{36d}\right].$$

Proof of theorems. The proof of the theorems (2.1) to (2.3) are quite straight forward.

For this, in order to prove Theorem 2.1, if we take $a = \frac{1}{2}, b = \frac{5}{2}$ and $c = 5$ in (1.19), we get after some simplification, the desired result of Theorem 2.1.

In exactly the same manner, the other theorems can be proved. □

3. Special cases

In this section, we shall mention a large number of new and interesting series identities for $\frac{1}{\Pi}$ from our main results.

(a) Results obtained from Theorem 2.1. For $d = 5$, we obtain the result

$$(3.1)\qquad {}_2F_1\left[\begin{matrix}\frac{1}{2}, & \frac{5}{2} \\ & 5\end{matrix}\;;\;1\right] = \frac{512}{105\pi},$$

which is equivalent to the result (1.7) of Liu, et al. [17].

(b) Results obtained from Theorem 2.2.

(1) For $d = 3$ in (2.2), we obtain the result

$$(3.2)\qquad {}_2F_1\left[\begin{matrix}\frac{1}{2}, & \frac{3}{2} \\ & 3\end{matrix}\;;\;1\right] = \frac{16}{3\pi},$$

which is equivalent to the result (1.6) of Liu, et al. [17].

(2) For $d = 3$ in (2.3), we obtain the result

$$ {}_2F_1\left[\begin{matrix} \frac{1}{3}, & \frac{5}{3} \\ & 3 \end{matrix} ; 1\right] = \frac{27\sqrt{3}}{10\pi}. \tag{3.3} $$

which is equivalent to the result (1.8) of Liu, et al. [17].

(3) For $d = 3$ in (2.4), we obtain the result

$$ {}_2F_1\left[\begin{matrix} \frac{2}{3}, & \frac{4}{3} \\ & 3 \end{matrix} ; 1\right] = \frac{27\sqrt{3}}{8\pi}. \tag{3.4} $$

which is equivalent to the result (1.9) of Liu, et al. [17].

(4) For $d = 3$ in (2.5), we obtain the result

$$ {}_2F_1\left[\begin{matrix} \frac{1}{6}, & \frac{11}{6} \\ & 3 \end{matrix} ; 1\right] = \frac{216}{55\pi}. \tag{3.5} $$

which is equivalent to the result (1.10) of Liu, et al. [17].

(5) For $d = 3$ in (2.6), we obtain the result

$$ {}_2F_1\left[\begin{matrix} \frac{5}{6}, & \frac{7}{6} \\ & 3 \end{matrix} ; 1\right] = \frac{216}{35\pi}. \tag{3.6} $$

which is equivalent to the result (1.11) of Liu, et al. [17].

(6) For $d = 3$ in (2.7), we obtain the result

$$ {}_2F_1\left[\begin{matrix} \frac{1}{4}, & \frac{7}{4} \\ & 3 \end{matrix} ; 1\right] = \frac{64\sqrt{2}}{21\pi}. \tag{3.7} $$

which is equivalent to the result (1.12) of Liu, et al. [17].

(7) For $d = 3$ in (2.8), we obtain the result

$$ {}_2F_1\left[\begin{matrix} \frac{3}{4}, & \frac{5}{4} \\ & 3 \end{matrix} ; 1\right] = \frac{64\sqrt{2}}{15\pi}. \tag{3.8} $$

which is equivalent to the result (1.13) of Liu, et al. [17].

(c) Results obtained from Theorem 2.3.

(1) For $d = 2$ in (2.9), we obtain the result

$$ {}_2F_1\left[\begin{matrix} \frac{1}{2}, & \frac{1}{2} \\ & 2 \end{matrix} ; 1\right] = \frac{4}{\pi}. \tag{3.9} $$

which is equivalent to the result (1.14) of Liu, et al. [17].

(2) For $d = 2$ in (2.10), we obtain the result

$$ {}_2F_1\left[\begin{matrix} \frac{1}{3}, & \frac{2}{3} \\ & 2 \end{matrix} ; 1\right] = \frac{9\sqrt{3}}{4\pi}. \tag{3.10}$$

which is equivalent to the result (1.15) of Liu, et al. [17].

(3) For $d = 2$ in (2.11), we obtain the result

$$ {}_2F_1\left[\begin{matrix} \frac{3}{4}, & \frac{1}{4} \\ & 2 \end{matrix} ; 1\right] = \frac{8\sqrt{2}}{3\pi}. \tag{3.11}$$

which is equivalent to the result (1.16) of Liu, et al. [17].

(4) For $d = 2$ in (2.12), we obtain the result

$$ {}_2F_1\left[\begin{matrix} \frac{1}{6}, & \frac{5}{6} \\ & 2 \end{matrix} ; 1\right] = \frac{18}{5\pi}. \tag{3.12}$$

which is equivalent to the result (1.17) of Liu, et al. [17].

All the previous special cases can also be found in [18, Table-1, P. 4]

4. Concluding remarks

In this research paper, first we have established twelve general series identities for $\frac{1}{\pi}$ by employing extension of classical Gauss's summation theorem and afterwards, deduced a large number of elementary and new series identities for $\frac{1}{\pi}$.

By rewriting (1.19) in the following form

$$ {}_3F_2\left[\begin{matrix} a, & b, & d+1 \\ & c+1, & d \end{matrix} ; 1\right] $$
$$ = \frac{c}{(c-a)(c-b)}\left[c-a-b+\frac{ab}{d}\right] {}_2F_1\left[\begin{matrix} a, & b \\ & c \end{matrix} ; 1\right] $$
$$ = \frac{c}{(c-a)(c-b)}\left[c-a-b+\frac{ab}{d}\right] \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{4.1}$$

provided $\mathrm{Re}(c-a-b) > 0$ and $d \neq 0, -1, -2, \ldots$, several other results for multiples of Π, Π^2, $\frac{1}{\Pi^2}$, $\frac{1}{\Pi^3}$ and $\frac{1}{\sqrt{\Pi}}$ can be obtained. For example:

(1) If in (4.1), we put $c = \frac{7}{2}$, $a = \frac{1}{2}$ and $b = \frac{3}{2}$, we have

$$ {}_3F_2\left[\begin{matrix} \frac{1}{2}, & \frac{3}{2}, & d+1 \\ & \frac{9}{2}, & d \end{matrix} ; 1\right] = \frac{105\Pi}{256}\left[1+\frac{1}{2d}\right] $$

and by replacing d by $\frac{7}{2}$, we get

$$ {}_2F_1\left[\begin{matrix} \frac{1}{2}, & \frac{3}{2} \\ & \frac{7}{2} \end{matrix} ; 1\right] = \frac{15}{32}\Pi. \tag{4.2} $$

(2) If in (4.1), we put $c = \frac{9}{4}$, $a = \frac{1}{2}$ and $b = \frac{3}{2}$, we have

$$ {}_3F_2\left[\begin{matrix} \frac{1}{2}, & \frac{3}{2}, & d+1 \\ & \frac{13}{4}, & d \end{matrix} ; 1\right] = \frac{5\Pi^2}{56\Gamma^4\left(\frac{3}{4}\right)}\left[1 + \frac{3}{d}\right] $$

and by replacing d by $\frac{9}{4}$, we get

$$ {}_2F_1\left[\begin{matrix} \frac{1}{2}, & \frac{3}{2} \\ & \frac{9}{4} \end{matrix} ; 1\right] = \frac{5}{24\Gamma^4\left(\frac{3}{4}\right)}\Pi^2. \tag{4.3} $$

(3) If in (4.1), we put $c = \frac{13}{6}$, $a = \frac{1}{2}$ and $b = \frac{2}{3}$, we have

$$ {}_3F_2\left[\begin{matrix} \frac{1}{2}, & \frac{2}{3}, & d+1 \\ & \frac{19}{6}, & d \end{matrix} ; 1\right] = \frac{91\Gamma^3\left(\frac{1}{3}\right)}{120\sqrt[3]{2}\ \Pi^2}\left[1 + \frac{1}{3d}\right] $$

and by replacing d by $\frac{3}{16}$, we get

$$ {}_2F_1\left[\begin{matrix} \frac{1}{2}, & \frac{2}{3} \\ & \frac{3}{16} \end{matrix} ; 1\right] = \frac{455\Gamma^3\left(\frac{1}{3}\right)}{216\sqrt[3]{2}}\frac{1}{\Pi^2}. \tag{4.4} $$

(4) If in (4.1), we put $c = \frac{7}{6}$, $a = \frac{1}{3}$ and $b = \frac{1}{2}$, we have

$$ {}_3F_2\left[\begin{matrix} \frac{1}{3}, & \frac{1}{2}, & d+1 \\ & \frac{13}{6}, & d \end{matrix} ; 1\right] = \frac{21\sqrt{3}\Gamma^4\left(\frac{1}{3}\right)}{240(\sqrt[3]{2})^2\ \Pi^3}\left[1 + \frac{1}{2d}\right] $$

and by replacing d by $\frac{7}{6}$, we get

$$ {}_2F_1\left[\begin{matrix} \frac{1}{3}, & \frac{1}{2} \\ & \frac{7}{6} \end{matrix} ; 1\right] = \frac{\sqrt{3}\Gamma^4\left(\frac{1}{3}\right)}{8(\sqrt[3]{2})^2}\frac{1}{\Pi^3}. \tag{4.5} $$

(5) If in (4.1), we put $c = \frac{7}{4}$, $a = \frac{1}{4}$ and $b = \frac{3}{4}$, we have

$$ {}_3F_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4}, & d+1 \\ & \frac{11}{4}, & d \end{matrix} ; 1\right] = \frac{21\Gamma^2\left(\frac{3}{4}\right)}{16\sqrt{\Pi}}\left[1 + \frac{1}{4d}\right] $$

and by replacing d by $\frac{7}{4}$, we get

$$ {}_2F_1\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ & \frac{7}{4} \end{matrix} ; 1\right] = \frac{3\Gamma^2\left(\frac{3}{4}\right)}{2}\frac{1}{\sqrt{\Pi}}. \tag{4.6} $$

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