Commun. Korean Math. Soc. **32** (2017), No. 4, pp. 979–990
https://doi.org/10.4134/CKMS.c170012
pISSN: 1225-1763 / eISSN: 2234-3024

EVALUATION OF A NEW CLASS OF DOUBLE DEFINITE INTEGRALS

Sébastien Gaboury and Arjun Kumar Rathie

Abstract. Inspired by the results obtained by Brychkov ([2]), the authors evaluate a large number of new and interesting double definite integrals. The results are obtained with the use of classical hypergeometric summation theorems and a well-known double finite integral due to Edwards ([3]). The results are given in terms of Psi and Hurwitz zeta functions suitable for numerical computations.

1. Introduction

The generalized hypergeometric function ${}_pF_q$ with p numerator parameters $a_1, \ldots, a_p$ such that $a_j \in \mathbb{C}$ $(j = 1, \ldots, p)$ and q denominator parameters $b_1, \ldots, b_q$ such that $b_j \in \mathbb{C}\setminus\mathbb{Z}_0^-$ $(j = 1, \ldots, q;\ \mathbb{Z}_0^- := \mathbb{Z}\cup\{0\} = \{0, -1, -2, \ldots\})$ is defined by (see, for example [5, Chapter 4]; see also [7, pp. 71–72])

$$
\begin{aligned}
{}_pF_q\left[\begin{matrix}\alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q;\end{matrix}\ z\right] &= {}_pF_q[\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z] \\
&= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!}, \qquad (1.1)
\end{aligned}
$$

$(p \le q$ and $|z| < \infty;\ p = q + 1$ and $|z| < 1;\ p = q + 1,\ |z| = 1$ and $\Re(\omega) > 0)$ where

$$
\omega := \sum_{j=1}^{q} b_i - \sum_{j=1}^{p} a_i
$$

and $(\alpha)_n$ denotes the Pochhammer symbol defined, in terms of the Gamma function, by

$$
(\alpha)_n := \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)} = \begin{cases} \alpha(\alpha+1)\cdots(\alpha+n-1) & (n \in \mathbb{N}; \alpha \in \mathbb{C}) \\ 1 & (n = 0; \alpha \in \mathbb{C} \setminus \{0\}). \end{cases}
$$

Received January 12, 2017; Accepted March 17, 2017.
2010 *Mathematics Subject Classification.* Primary 33C20, 33C10.
Key words and phrases. hypergeometric functions, integrals, special functions, summation theorems.

It is well-known that, whenever a generalized hypergeometric function reduces to Gamma function, the results are very important from the application point of view. Therefore, the classical summation theorems such as those of Watson, Dixon and Whipple for the series ${}_3F_2$ play an important role. These are given respectively by [1, 5, 6]:

$$\begin{aligned}
&{}_3F_2\left[\begin{matrix} a, \quad b \quad , & c; \\ \frac{1}{2}(a+b+1) \quad , & 2c; \end{matrix}\; 1\right] \\
&= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)}
\end{aligned} \tag{1.2}$$

provided that $\Re(2c-a-b) > -1$,

$$\begin{aligned}
&{}_3F_2\left[\begin{matrix} a, \quad b \quad , & c; \\ 1+a-b \quad , & 1+a-c; \end{matrix}\; 1\right] \\
&= \frac{\Gamma\left(1+\frac{1}{2}a\right)\Gamma\left(1+a-b\right)\Gamma\left(1+a-c\right)\Gamma\left(1+\frac{1}{2}a-b-c\right)}{\Gamma\left(1+a\right)\Gamma\left(1+\frac{1}{2}a-b\right)\Gamma\left(1+\frac{1}{2}a-c\right)\Gamma\left(1+a-b-c\right)}
\end{aligned} \tag{1.3}$$

provided that $\Re(a-2b-2c) > -2$, and finally

$$\begin{aligned}
&{}_3F_2\left[\begin{matrix} a, \quad 1-a \quad , & c; \\ e \quad , & 1+2c-e; \end{matrix}\; 1\right] \\
&= \frac{2^{1-2c}\,\pi\,\Gamma(e)\Gamma(1+2c-e)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}e\right)\Gamma\left(\frac{1}{2}e-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}e+\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}e-\frac{1}{2}a+1\right)}
\end{aligned} \tag{1.4}$$

provided that $\Re(c) > 0$ and $\Re(e) > 0$.

In the theory of generalized hypergeometric functions, there exist a large number of hypergeometric identities or summation formulas that can be expressed in terms of Gamma functions for generalized hypergeometric functions ${}_pF_q$ with specified values of the arguments, usually taken to be 1, -1 and $\frac{1}{2}$.

Also, it is evident that, for every such hypergeometric identity, we can easily evaluate a number of finite integrals involving hypergeometric functions and the logarithmic functions. Since the Gauss's hypergeometric function ${}_2F_1$ and the confluent hypergeometric function ${}_1F_1$ are the core of the special functions and almost all the commonly elementary functions can be obtained either as a special case or a limiting case, thus the integrals involving these functions play an important role.

In a very interesting and useful paper, Brychkov [2] evaluated some integrals of this type by employing some of the above mentioned classical summation theorems and discussed several interesting special cases.

Inspired by this work, the authors, in this paper, aim at evaluating a large number of double finite integrals involving hypergeometric functions and logarithmic functions. Several interesting special cases are also given. The results are obtained in terms of the Psi function and the Hurwitz zeta function which are more suitable for numerical computations.

The following definitions of the Psi function and the Hurwitz zeta function are also given in this section so that the paper may be self contained.

The Psi function $\Psi(z)$ which consists in the logarithmic derivative of the Gamma function is given by [4, 7]

$$\Psi(z) := \frac{\Gamma'(z)}{\Gamma(z)} \tag{1.5}$$

and the celebrated Hurwitz zeta function $\zeta(s,a)$ is defined as [4, 7]

$$\zeta(s,a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \quad a \neq 0, -1, -2, \ldots;\ \Re(s) > 1. \tag{1.6}$$

Also, we record here the following well-known result:

$$\frac{\partial^k}{\partial z^k} \ln\left\{\prod_{j=1}^{m} \Gamma(a_j + z)\right\} = \begin{cases} \sum_{j=1}^{m} \Psi(a_j+z) & k = 1, \\ (-1)^k (k-1)! \sum_{j=1}^{m} \zeta(k, a_j + z) & k \geq 2. \end{cases} \tag{1.7}$$

Finally, we recall an important result for the sequel due to Edwards [3]:

$$\int_0^1 \int_0^1 y^{\alpha}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{1-\alpha-\beta}\, \mathrm{d}x\, \mathrm{d}y = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{1.8}$$

provided that $\Re(\alpha) > 0$ and $\Re(\beta) > 0$.

It is worthy to note that making the change of variable $x \mapsto \frac{z-y}{y(z-1)}$ in equation (1.8), we obtain its simpler form as

$$\int_0^1 \int_0^y z^{\alpha-1}(1-z)^{\beta-2}\, \mathrm{d}z\, \mathrm{d}y = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{1.9}$$

provided that $\Re(\alpha) > 0$ and $\Re(\beta) > 1$.

2. Main results

In this section, we present eight definite integrals formulas involving the Gauss's hypergeometric function and the logarithmic function. The derivation of these results will be treated in Sections 3 and 4.

First integral formula

$$\begin{aligned} &\int_0^1 \int_0^y z^{c-1}(1-z)^{c-2} \ln^n\left(z - z^2\right)\ {}_2F_1\left[\begin{matrix} a,\ b; \\ \frac{1}{2}(a+b+1); \end{matrix}\ z\right] \mathrm{d}z\, \mathrm{d}y \\ &= \frac{2\,\pi\,\Gamma\left(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}b + \frac{1}{2}\right)} \cdot \sum_{r=0}^{n-1} \binom{n-1}{r} \frac{\partial^r}{\partial c^r} A \cdot \frac{\partial^{n-r-1}}{\partial c^{n-r-1}} B, \end{aligned} \tag{2.1}$$

where

$$A = \frac{2^{-2c}\,\Gamma(c)\Gamma\left(c - \frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}\right)}{\Gamma\left(c - \frac{1}{2}a + \frac{1}{2}\right)\Gamma\left(c - \frac{1}{2}b + \frac{1}{2}\right)}, \tag{2.2}$$

$$\frac{\partial}{\partial c}A = A\cdot B, \tag{2.3}$$

$$B = -2\ln 2 + \Psi(c) + \Psi\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right) - \Psi\left(c-\frac{1}{2}a+\frac{1}{2}\right) - \Psi\left(c-\frac{1}{2}b+\frac{1}{2}\right), \tag{2.4}$$

$$\frac{\partial^n}{\partial c^n}A = \sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\partial^r}{\partial c^r}A\cdot\frac{\partial^{n-r-1}}{\partial c^{n-r-1}}B \tag{2.5}$$

and

$$\frac{\partial^{n-r-1}}{\partial c^{n-r-1}}B = (-1)^{n-r}(n-r-1)!\Bigg\{\zeta(n-r,c) + \zeta\left(n-r, c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right) - \zeta\left(n-r, c-\frac{1}{2}a+\frac{1}{2}\right) - \zeta\left(n-r, c-\frac{1}{2}b+\frac{1}{2}\right)\Bigg\}. \tag{2.6}$$

Second integral formula

$$\int_0^1\int_0^y z^{b-1}(1-z)^{\frac{1}{2}(a-b-1)-1}\ln^n\left(\frac{z}{\sqrt{1-z}}\right)\ {}_2F_1\left[\begin{matrix} a,\ c; \\ 2c; \end{matrix}\ z\right]\mathrm{d}z\ \mathrm{d}y$$
$$= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)}\cdot\sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\partial^r}{\partial b^r}A\cdot\frac{\partial^{n-r-1}}{\partial b^{n-r-1}}B, \tag{2.7}$$

where

$$A = \frac{\Gamma(b)\Gamma\left(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}, \tag{2.8}$$

$$\frac{\partial}{\partial b}A = A\cdot B, \tag{2.9}$$

$$B = \Psi(b) - \frac{1}{2}\Psi\left(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right) - \frac{1}{2}\Psi\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right) - \frac{1}{2}\Psi\left(\frac{1}{2}b+\frac{1}{2}\right) + \frac{1}{2}\Psi\left(c-\frac{1}{2}b+\frac{1}{2}\right), \tag{2.10}$$

$$\frac{\partial^n}{\partial b^n}A = \sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\partial^r}{\partial b^r}A\cdot\frac{\partial^{n-r-1}}{\partial b^{n-r-1}}B \tag{2.11}$$

and

$$\frac{\partial^{n-r-1}}{\partial b^{n-r-1}}B = (-1)^{n-r}(n-r-1)!\Bigg\{\zeta(n-r,b) - \frac{(-1)^{n-r-1}}{2^{n-r}}\zeta\left(n-r,\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)$$

$$
\begin{aligned}
&- \frac{(-1)^{n-r-1}}{2^{n-r}}\zeta\left(n-r, c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right) \\
&- \frac{1}{2^{n-r}}\zeta\left(n-r, \frac{1}{2}b+\frac{1}{2}\right) \\
&+ \frac{(-1)^{n-r-1}}{2^{n-r}}\zeta\left(n-r, c-\frac{1}{2}b+\frac{1}{2}\right)\Bigg\}. \qquad (2.12)
\end{aligned}
$$

Third integral formula

$$
\begin{aligned}
&\int_0^1\int_0^y z^{c-1}(1-z)^{c-2}\ln^n\left(z-z^2\right)\,{}_2F_1\left[\begin{matrix} a,\ b; \\ \frac{1}{2}(a+b+1); \end{matrix}\ 1-z\right]\mathrm{d}z\,\mathrm{d}y \\
&= \frac{2\,\pi\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}\cdot\sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\partial^r}{\partial c^r}A\cdot\frac{\partial^{n-r-1}}{\partial c^{n-r-1}}B, \qquad (2.13)
\end{aligned}
$$

where the values of A, B and their derivatives are the same as given in (2.2) to (2.6).

Fourth integral formula

$$
\begin{aligned}
&\int_0^1\int_0^y z^{\frac{1}{2}(1+a-b)-1}(1-z)^{b-2}\ln^n\left(\frac{1-z}{\sqrt{z}}\right)\,{}_2F_1\left[\begin{matrix} a,\ c; \\ 2c; \end{matrix}\ 1-z\right]\mathrm{d}z\,\mathrm{d}y \\
&= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)}\cdot\sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\partial^r}{\partial b^r}A\cdot\frac{\partial^{n-r-1}}{\partial b^{n-r-1}}B, \qquad (2.14)
\end{aligned}
$$

where the values of A, B and their derivatives are the same as given in (2.2) to (2.6).

Fifth integral formula

$$
\begin{aligned}
&\int_0^1\int_0^y z^{b-1}(1-z)^{a-2b-1}\ln^n\left(\frac{z}{(1-z)^2}\right)\,{}_2F_1\left[\begin{matrix} a,\ c; \\ 1+a-c; \end{matrix}\ z\right]\mathrm{d}z\,\mathrm{d}y \\
&= \frac{\Gamma\left(1+a-c\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(1+\frac{1}{2}a-c\right)}\cdot\sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\partial^r}{\partial b^r}A\cdot\frac{\partial^{n-r-1}}{\partial b^{n-r-1}}B, \qquad (2.15)
\end{aligned}
$$

where

$$
A = \frac{2^{-2b}\,\Gamma(b)\Gamma\left(\frac{1}{2}a-b+\frac{1}{2}\right)\Gamma\left(1+\frac{1}{2}a-b-c\right)}{\Gamma\left(1+a-b-c\right)}, \qquad (2.16)
$$

$$
\frac{\partial}{\partial b}A = A\cdot B, \qquad (2.17)
$$

$$
\begin{aligned}
B = &-2\ln 2+\Psi(b)-\Psi\left(\frac{1}{2}a-b+\frac{1}{2}\right)-\Psi\left(1+\frac{1}{2}a-b-c\right) \\
&-\Psi\left(1+a-b-c\right), \qquad (2.18)
\end{aligned}
$$

$$\frac{\partial^n}{\partial b^n}A=\sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\partial^r}{\partial b^r}A\cdot\frac{\partial^{n-r-1}}{\partial b^{n-r-1}}B \tag{2.19}$$

and

$$\frac{\partial^{n-r-1}}{\partial b^{n-r-1}}B=(n-r-1)!\Bigg\{(-1)^{n-r}\zeta(n-r,b)+\zeta\left(n-r,\frac{1}{2}a-b+\frac{1}{2}\right)$$
$$+\zeta\left(n-r,1+\frac{1}{2}a-b-c\right)-\zeta\left(n-r,1+a-b-c\right). \tag{2.20}$$

Sixth integral formula

$$\int_0^1\int_0^y z^{a-2b}(1-z)^{b-2}\ln^n\left(\frac{1-z}{z^2}\right)\ {}_2F_1\left[\begin{matrix}a,\ c;\\ 1+a-c;\end{matrix}\ 1-z\right]\mathrm{d}z\ \mathrm{d}y$$
$$=\frac{\Gamma\left(1+a-c\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(1+\frac{1}{2}a-c\right)}\cdot\sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\partial^r}{\partial b^r}A\cdot\frac{\partial^{n-r-1}}{\partial b^{n-r-1}}B, \tag{2.21}$$

where the values of A, B and their derivatives are the same as given in (2.16) to (2.20).

Seventh integral formula

$$\int_0^1\int_0^y z^{c-1}(1-z)^{c-e-1}\ln^n\left(z-z^2\right)\ {}_2F_1\left[\begin{matrix}a,\ 1-a;\\ e;\end{matrix}\ z\right]\mathrm{d}z\ \mathrm{d}y$$
$$=\frac{2\,\pi\,\Gamma\left(e\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}e\right)\Gamma\left(\frac{1}{2}e-\frac{1}{2}a+\frac{1}{2}\right)}\cdot\sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\partial^r}{\partial c^r}A\cdot\frac{\partial^{n-r-1}}{\partial c^{n-r-1}}B, \tag{2.22}$$

where

$$A=\frac{2^{-2c}\,\Gamma(c)\Gamma\left(1-e+c\right)}{\Gamma\left(c-\frac{1}{2}e+\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}e-\frac{1}{2}a+1\right)}, \tag{2.23}$$

$$\frac{\partial}{\partial c}A=A\cdot B, \tag{2.24}$$

$$B=-\,2\ln 2+\Psi(c)+\Psi\left(1-e+c\right)-\Psi\left(c-\frac{1}{2}e+\frac{1}{2}a+\frac{1}{2}\right)$$
$$-\,\Psi\left(c-\frac{1}{2}e-\frac{1}{2}a+1\right), \tag{2.25}$$

$$\frac{\partial^n}{\partial c^n}A=\sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\partial^r}{\partial c^r}A\cdot\frac{\partial^{n-r-1}}{\partial c^{n-r-1}}B \tag{2.26}$$

and

$$\frac{\partial^{n-r-1}}{\partial c^{n-r-1}}B=(-1)^{n-r}(n-r-1)!\Bigg\{\zeta(n-r,c)+\zeta\left(n-r,1-e+c\right)$$

$$-\zeta\left(n-r, c-\frac{1}{2}e+\frac{1}{2}a+\frac{1}{2}\right)-\zeta\left(n-r, c-\frac{1}{2}e-\frac{1}{2}a+1\right)\Bigg\}. \tag{2.27}$$

Eighth integral formula

$$\int_0^1\int_0^y z^{c-e}(1-z)^{c-2}\ln^n\left(z-z^2\right)\ {}_2F_1\left[\begin{matrix} a,\ 1-a; \\ e; \end{matrix}\ 1-z\right]\mathrm{d}z\ \mathrm{d}y$$
$$= \frac{2\,\pi\,\Gamma\,(e)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}e\right)\Gamma\left(\frac{1}{2}e-\frac{1}{2}a+\frac{1}{2}\right)}\cdot\sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\partial^r}{\partial c^r}A\cdot\frac{\partial^{n-r-1}}{\partial c^{n-r-1}}B, \tag{2.28}$$

where the values of A, B and their derivatives are the same as given in (2.23) to (2.27).

3. Derivations of the results

In this section, in order to prove our new class of definite double integrals given in Section 2, we shall establish, first, eight definite double integrals involving hypergeometric functions. Next, we will use these formulas to prove our main results.

These eight definite double integrals involving hypergeometric functions are:

$$\int_0^1\int_0^y z^{c-1}(1-z)^{c-2}\ {}_2F_1\left[\begin{matrix} a,\ b; \\ \frac{1}{2}(a+b+1); \end{matrix}\ z\right]\mathrm{d}z\ \mathrm{d}y$$
$$= \frac{\pi\ 2^{1-2c}\,\Gamma\,(c)\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)} = I_1 \tag{3.1}$$

provided that $\Re(c) > 1$ and $\Re(2c-a-b) > -1$,

$$\int_0^1\int_0^y z^{b-1}(1-z)^{\frac{1}{2}(a-b-1)-1}\ {}_2F_1\left[\begin{matrix} a,\ c; \\ 2c; \end{matrix}\ z\right]\mathrm{d}z\ \mathrm{d}y$$
$$= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(b)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)} = I_2 \tag{3.2}$$

provided that $\Re(b) > 0$ and $\Re(a-b-1) > 0$,

$$\int_0^1\int_0^y z^{c-1}(1-z)^{c-2}\ {}_2F_1\left[\begin{matrix} a,\ b; \\ \frac{1}{2}(a+b+1); \end{matrix}\ 1-z\right]\mathrm{d}z\ \mathrm{d}y = I_1, \tag{3.3}$$

$$\int_0^1\int_0^y z^{\frac{1}{2}(1+a-b)-1}(1-z)^{b-2}\ {}_2F_1\left[\begin{matrix} a,\ c; \\ 2c; \end{matrix}\ 1-z\right]\mathrm{d}z\ \mathrm{d}y = I_2 \tag{3.4}$$

provided that $\Re(b) > 1$ and $\Re(1+a-b) > 0$,

$$\int_0^1\int_0^y z^{b-1}(1-z)^{a-2b-1}\ {}_2F_1\left[\begin{matrix} a,\ c; \\ 1+a-c; \end{matrix}\ z\right]\mathrm{d}z\ \mathrm{d}y$$
$$= \frac{2^{-2b}\,\Gamma\,(b)\,\Gamma\,(1+a-c)\,\Gamma\left(\frac{1}{2}a-b+\frac{1}{2}\right)\Gamma\left(1+\frac{1}{2}a-b-c\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(1+\frac{1}{2}a-c\right)\Gamma\left(1+a-b-c\right)} = I_3 \tag{3.5}$$

provided that $\Re(b) > 0$ and $\Re(a - 2b) > 0$,

$$\int_0^1 \int_0^y z^{a-2b}(1-z)^{b-2} \, {}_2F_1\left[\begin{matrix} a, \ c; \\ 1+a-c; \end{matrix} \ 1-z\right] \mathrm{d}z \, \mathrm{d}y = I_3 \tag{3.6}$$

provided that $\Re(b) > 1$ and $\Re(a - 2b) > 0$,

(3.7)

$$\int_0^1 \int_0^y z^{c-1}(1-z)^{c-e-1} \, {}_2F_1\left[\begin{matrix} a, \ 1-a; \\ e; \end{matrix} \ z\right] \mathrm{d}z \, \mathrm{d}y$$

$$= \frac{\pi \, 2^{1-2c} \, \Gamma(c) \, \Gamma(e) \, \Gamma(1-e+c)}{\Gamma\left(\frac{1}{2}a + \frac{1}{2}e\right) \Gamma\left(\frac{1}{2}e - \frac{1}{2}a + \frac{1}{2}\right) \Gamma\left(c - \frac{1}{2}e + \frac{1}{2}a + \frac{1}{2}\right) \Gamma\left(c - \frac{1}{2}e - \frac{1}{2}a + 1\right)} = I_4$$

provided that $\Re(c) > 0$ and $\Re(e) > 0$,

$$\int_0^1 \int_0^y z^{c-e}(1-z)^{c-2} \, {}_2F_1\left[\begin{matrix} a, \ 1-a; \\ e; \end{matrix} \ 1-z\right] \mathrm{d}z \, \mathrm{d}y = I_4 \tag{3.8}$$

provided that $\Re(c) > 1$ and $\Re(c - e + 1) > 0$.

Let us consider the first integral formula (3.1). Denoting the left-hand side of (3.1) by I, expressing the hypergeometric function ${}_2F_1$ as a series, changing the order of integration and summation (which is easily seen to be justified due to the uniform convergence of the involved series), evaluating the double integral with the help of (1.9) and after some elementary simplifications, summing up the series, we get

$$I = \frac{\{\Gamma(c)\}^2}{\Gamma(2c)} \, {}_3F_2\left[\begin{matrix} a, \ b \ , \ c; \\ \frac{1}{2}(a+b+1) \ , \ 2c; \end{matrix} \ 1\right]. \tag{3.9}$$

It is easily seen that the ${}_3F_2$ can be evaluated with the help of the Watson's summation formula (1.2) and after some simplifications, we arrive at the right-hand side of (3.1).

Proceeding exactly in the same manner, the remaining integrals (3.2) to (3.8) can also be evaluated by using appropriate summation theorems (1.2) to (1.4).

We can now proceed to the proofs of our main results presented in Section 2. Let us start with the integral formula (3.1). If we differentiate n-times both sides of (3.1) with respect to c, we get

$$\int_0^1 \int_0^y z^{c-1}(1-z)^{c-2} \ln^n\left(z - z^2\right) \, {}_2F_1\left[\begin{matrix} a, \ b; \\ \frac{1}{2}(a+b+1); \end{matrix} \ z\right] \mathrm{d}z \, \mathrm{d}y$$

$$= \frac{2 \, \pi \, \Gamma\left(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}b + \frac{1}{2}\right)} \cdot \frac{\partial^n}{\partial c^n} A, \tag{3.10}$$

where A is the same as given in (2.2). We easily see that

$$\frac{\partial}{\partial c} A = A \, \frac{\partial}{\partial c} \ln A = A \cdot B, \tag{3.11}$$

where B is the same as given in (2.4). Now, from (3.11), we have

$$\frac{\partial^n}{\partial c^n}A = \frac{\partial^{n-1}}{\partial c^{n-1}}\left\{\frac{\partial}{\partial c}A\right\} = \frac{\partial^{n-1}}{\partial c^{n-1}}\left\{A \cdot B\right\}. \tag{3.12}$$

Using the well-known Leibniz theorem, this becomes

$$\frac{\partial^n}{\partial c^n}A = \sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\partial^r}{\partial c^r}A \cdot \frac{\partial^{n-r-1}}{\partial c^{n-r-1}}B, \tag{3.13}$$

where $\frac{\partial^{n-r-1}}{\partial c^{n-r-1}}B$ is the same as given in (2.6).

Finally, substituting the values of $\frac{\partial^n}{\partial c^n}A$ from (3.13) into (3.10), we obtain the result asserted in (2.1).

In exactly the same way, the other seven results presented in Section 2 can also be proved from the integrals (3.2) to (3.8) respectively.

4. Special cases

This section is devoted to present some interesting special cases of our eight definite double integrals.

Setting $a = b = 1$ in (2.1) and appealing to the following well-known result [4, p. 476, Equ. (147)]

$${}_2F_1\left[\begin{matrix} 1,\ 1; \\ \frac{3}{2}; \end{matrix}\ x\right] = \frac{\arcsin\sqrt{x}}{\sqrt{x(1-x)}}, \tag{4.1}$$

we find the following result

$$\int_0^1\int_0^y z^{c-\frac{3}{2}}(1-z)^{c-\frac{5}{2}}\ln^n\left(z-z^2\right)\ \arcsin\left(\sqrt{z}\right)\mathrm{d}z\,\mathrm{d}y$$
$$= \pi^{\frac{3}{2}} \cdot \sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\partial^r}{\partial c^r}A \cdot \frac{\partial^{n-r-1}}{\partial c^{n-r-1}}B \tag{4.2}$$

for $\Re(c) > \frac{3}{2}$ and

$$A = \frac{2^{-2c}\,\Gamma\left(c-\frac{1}{2}\right)}{\Gamma\left(c\right)}, \tag{4.3}$$

$$\frac{\partial}{\partial c}A = A \cdot B, \tag{4.4}$$

$$B = -2\ln 2 - \Psi(c) + \Psi\left(c - \frac{1}{2}\right), \tag{4.5}$$

$$\frac{\partial^n}{\partial c^n}A = \sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\partial^r}{\partial c^r}A \cdot \frac{\partial^{n-r-1}}{\partial c^{n-r-1}}B \tag{4.6}$$

and

$$\frac{\partial^{n-r-1}}{\partial c^{n-r-1}} B = (-1)^{n-r}(n-r-1)!\left\{\zeta\left(n-r, c-\frac{1}{2}\right) - \zeta\left(n-r, c\right)\right\}. \tag{4.7}$$

If we let $b = -a$ in (2.1) and using the following result [4, p. 459, Equ. (105)]

$$ {}_2F_1\left[\begin{array}{c} a, \ -a; \\ \frac{1}{2}; \end{array} x\right] = \cos\left(2a \arcsin\sqrt{x}\right), \tag{4.8}$$

we have

$$\begin{aligned} &\int_0^1\int_0^y z^{c-1}(1-z)^{c-2}\ln^n\left(z-z^2\right)\ \cos\left(2a\arcsin\left(\sqrt{z}\right)\right) \mathrm{d}z\ \mathrm{d}y \\ &= \frac{2\ \pi^{\frac{3}{2}}}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}a\right)} \cdot \sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\partial^r}{\partial c^r}A \cdot \frac{\partial^{n-r-1}}{\partial c^{n-r-1}}B \end{aligned} \tag{4.9}$$

for $\Re(c) > 1$ and

$$A = \frac{2^{-2c}\ \Gamma(c)\Gamma\left(c+\frac{1}{2}\right)}{\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}a+\frac{1}{2}\right)}, \tag{4.10}$$

$$\frac{\partial}{\partial c}A = A \cdot B, \tag{4.11}$$

(4.12)

$$B = -2\ln 2 + \Psi(c) + \Psi\left(c+\frac{1}{2}\right) - \Psi\left(c-\frac{1}{2}a+\frac{1}{2}\right) - \Psi\left(c+\frac{1}{2}a+\frac{1}{2}\right),$$

$$\frac{\partial^n}{\partial c^n}A = \sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\partial^r}{\partial c^r}A \cdot \frac{\partial^{n-r-1}}{\partial c^{n-r-1}}B \tag{4.13}$$

and

$$\begin{aligned} \frac{\partial^{n-r-1}}{\partial c^{n-r-1}}B = (-1)^{n-r}(n-r-1)!&\left\{\zeta(n-r,c) + \zeta\left(n-r, c+\frac{1}{2}\right)\right. \\ &\left. - \zeta\left(n-r, c-\frac{1}{2}a+\frac{1}{2}\right) - \zeta\left(n-r, c+\frac{1}{2}a+\frac{1}{2}\right)\right\}. \end{aligned} \tag{4.14}$$

Letting $b = -2m-1$ in (2.1) and replace a by $a+2m+1$, where m is zero or a positive integer, we get the following elegant result

(4.15)

$$\int_0^1\int_0^y z^{c-1}(1-z)^{c-2}\ln^n\left(z-z^2\right)\ {}_2F_1\left[\begin{array}{c} -2m-1, \ a+2m+1; \\ \frac{1}{2}(a+1); \end{array} z\right] \mathrm{d}z\ \mathrm{d}y = 0$$

for $\Re(c) > 1$.

Let us see a last special case. Putting $a = c = \frac{1}{2}$ in (2.7) and making use of the following result [4, p. 473, Equ. (75)]

$$ {}_2F_1\left[\begin{matrix} \frac{1}{2}, \frac{1}{2}; \\ 1; \end{matrix} \; x\right] = \frac{2}{\pi}\, K\left(\sqrt{x}\right), \tag{4.16} $$

where $K(x)$ holds for the complete elliptic integral of the first kind, we find

$$ \int_0^1\int_0^y z^{b-1}(1-z)^{\frac{1}{2}(a-b-1)-1}\ln^n\left(\frac{z}{\sqrt{1-z}}\right)\, K\left(\sqrt{z}\right) \mathrm{d}z\, \mathrm{d}y $$
$$ = \frac{\pi^{\frac{3}{2}}}{2\,\left\{\Gamma\left(\frac{3}{4}\right)\right\}^2}\cdot\sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\partial^r}{\partial b^r}A\cdot\frac{\partial^{n-r-1}}{\partial b^{n-r-1}}B, \tag{4.17} $$

where

$$ A = \frac{\Gamma(b)\left\{\Gamma\left(\frac{3}{4}-\frac{1}{2}b\right)\right\}^2}{\Gamma\left(1-\frac{1}{2}b\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}, \tag{4.18} $$

$$ \frac{\partial}{\partial b}A = A\cdot B, \tag{4.19} $$

$$ B = \Psi(b) - \Psi\left(\frac{3}{4}-\frac{1}{2}b\right) - \frac{1}{2}\Psi\left(\frac{1}{2}b+\frac{1}{2}\right) + \frac{1}{2}\Psi\left(1-\frac{1}{2}b\right), \tag{4.20} $$

$$ \frac{\partial^n}{\partial b^n}A = \sum_{r=0}^{n-1}\binom{n-1}{r}\frac{\partial^r}{\partial b^r}A\cdot\frac{\partial^{n-r-1}}{\partial b^{n-r-1}}B \tag{4.21} $$

and

$$ \frac{\partial^{n-r-1}}{\partial b^{n-r-1}}B = (-1)^{n-r}(n-r-1)!\Bigg\{\zeta(n-r,b) - \frac{(-1)^{n-r-1}}{2^{n-r-1}}\zeta\left(n-r,\frac{3}{4}-\frac{1}{2}b\right) $$
$$ - \frac{1}{2^{n-r}}\zeta\left(n-r,\frac{1}{2}b+\frac{1}{2}\right) + \frac{(-1)^{n-r-1}}{2^{n-r}}\zeta\left(n-r,1-\frac{1}{2}b\right)\Bigg\}. \tag{4.22} $$

Remark 1. From integrals (2.13) and (2.14), we can obtain special cases similar to the ones given above. So, we prefer to avoid the details.

Remark 2. Similarly by using the special cases of ${}_2F_1$ given above and also by the following known results [4, p. 469, Equ. (17); p. 472, Equ. (65); p. 473, Equ. (91); p. 473, Equ. (83)]

$$ {}_2F_1\left[\begin{matrix} -\frac{1}{2}, \frac{3}{2}; \\ 1; \end{matrix} \; x\right] = \frac{2}{\pi}\left[K\left(\sqrt{x}\right) - \frac{K\left(\sqrt{x}\right)-E\left(\sqrt{x}\right)}{x}\right], \tag{4.23} $$

$$ {}_2F_1\left[\begin{matrix} \frac{1}{4}, \frac{3}{4}; \\ \frac{3}{2}; \end{matrix} \; x\right] = \sqrt{2}\,\left[1+\sqrt{1-x}\right]^{-\frac{1}{2}}, \tag{4.24} $$

$$ {}_2F_1\left[\begin{matrix} \frac{1}{2}, \frac{3}{2}; \\ 2; \end{matrix} \; x\right] = \frac{4}{\pi}\left(\frac{K\left(\sqrt{x}\right)-E\left(\sqrt{x}\right)}{x}\right) \tag{4.25} $$

and

$$ {}_2F_1\left[\begin{matrix} 1, \frac{1}{2}; \\ \frac{3}{2}; \end{matrix}\ x\right] = \frac{1}{2\sqrt{x}}\ \ln\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right), \tag{4.26} $$

where $E(x)$ denotes the complete elliptic integral of the second kind, various other interesting special cases for the integrals (2.15), (2.21), (2.22) and (2.28) can be obtained.

References

[1] W. N. Bailey, *Generalized Hypergeometric Series*, Cambridge Math. Tracts, vol. 32. Cambridge University Press, Cambridge (1935). Reprinted by Stechert-Hafner, New York, 1964.

[2] Yu. A. Brychkov, *Evaluation of some classes of definite and indefinite integrals*, Integral Transforms Spec. Funct. **13** (2002), no. 2, 163–167.

[3] J. Edwards, *A treatise on the integral calculus with applications, examples and problems II*, Chelsea Publishing Company, New York, 1954.

[4] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Intregrals and series, Vol. 3*: *More special functions*, Gordon and Breach Science Publishers, New York, 1990.

[5] E. D. Rainville, *Special Functions*, Macmillan Company, New York, 1960.

[6] L. J. Slater, *Generalized Hypergeometric Functions*, Cambridge University Press London, 1966.

[7] H. M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science Publishers, Amsterdam, London and New York, 2012.

Sébastien Gaboury
Department of Mathematics and Computer Science
University of Québec at Chicoutimi
Chicoutimi, Québec G7H 2B1, Canada
E-mail address: s1gabour@uqac.ca

Arjun Kumar Rathie
Department of Mathematics
School of Mathematical and Physical Sciences
Central University of Kerala
Riverside Transit Campus, Padennakad
P. O. Nileshwar, Kasaragod 671 328, Kerala, India
E-mail address: akrathie@cukerala.ac.in