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TWO GENERAL RESULTS INVOLVING SQUARE OF GENERALIZED HYPERGEOMETRIC SERIES AND A COMPANION RESULT

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Abstract

In the theory of generalized hypergeometric functions ${}_pF_q$, among other things, summation, transformation, and product formulas are important. In the literature, there exist only a few results involving square of generalized hypergeometric functions. The objective of this

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paper is to provide explicit expressions of three types of $({}_1F_1)^2$ in the most general form. The results presented here are derived with the help of generalizations of Kummer's second theorem. We also consider some special cases of the main results given here, one of which is pointed out to be reduced to a known identity.

1. Introduction and Preliminaries

We begin by recalling the following well known and useful transformation formulas due to Kummer [4]:

$$e^{-x}{}_1F_1\left[\begin{matrix} a; \\ b; \end{matrix}\; x\right] = {}_1F_1\left[\begin{matrix} b-a; \\ b; \end{matrix}\; -x\right] \tag{1.1}$$

and

$$e^{-x/2}{}_1F_1\left[\begin{matrix} a; \\ 2a; \end{matrix}\; x\right] = {}_0F_1\left[\begin{matrix} —; \\ a+\frac{1}{2}; \end{matrix}\; \frac{x^2}{16}\right]. \tag{1.2}$$

Bailey [1] proved (1.1) with the help of the classical Gauss's summation theorem (see, e.g., [2, 3, 7, 10]):

$${}_2F_1\left[\begin{matrix} a,\, b; \\ c; \end{matrix}\; 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (\Re(c-a-b) > 0) \tag{1.3}$$

and (1.2) by using the classical Gauss's second summation theorem (see, e.g., [2]):

$${}_2F_1\left[\begin{matrix} a,\, b; \\ \frac{1}{2}(a+b+1); \end{matrix}\; \frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}. \tag{1.4}$$

Rathie and Choi [9] derived the following formula which is equivalent to (1.2):

$${}_1F_1\left[\begin{matrix} a; \\ 2a; \end{matrix}\; 2x\right] = e^{x}{}_0F_1\left[\begin{matrix} —; \\ a+\frac{1}{2}; \end{matrix}\; \frac{x^2}{4}\right] \tag{1.5}$$

by employing the Gauss's summation theorem (1.3).

Two general results involving square of generalized hypergeometric series and a companion result

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From the theory of differential equations, Preece [5] obtained the following well known and very useful identity involving the product of the generalized hypergeometric functions:

$$ {}_1F_1\left[\begin{matrix} a; \\ 2a; \end{matrix}\; x\right] \cdot {}_1F_1\left[\begin{matrix} a; \\ 2a; \end{matrix}\; -x\right] = {}_1F_2\left[\begin{matrix} a; \\ a+\frac{1}{2},\; 2a; \end{matrix}\; \frac{x^2}{4}\right]. \tag{1.6} $$

Bailey [1] re-derived the identity (1.6) by using the following classical Watson's summation theorem for ${}_3F_2(1)$ (see, e.g., [10, p. 351]):

$$ {}_3F_2\left[\begin{matrix} a,\; b,\; c; \\ \frac{1}{2}(a+b+1),\; 2c; \end{matrix}\; 1\right] $$

$$ = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)}, \tag{1.7} $$

provided $\Re(2c - a - b) > -1$.

We use the Kummer's first transformation formula (1.1) to rewrite the Preece's identity (1.6) as follows:

$$ e^{-x}{}_1F_1\left[\begin{matrix} a; \\ 2a; \end{matrix}\; x\right] \cdot {}_1F_1\left[\begin{matrix} a; \\ 2a; \end{matrix}\; x\right] = {}_1F_2\left[\begin{matrix} a; \\ a+\frac{1}{2},\; 2a; \end{matrix}\; \frac{x^2}{4}\right], \tag{1.8} $$

or, equivalently,

$$ \left\{{}_1F_1\left[\begin{matrix} a; \\ 2a; \end{matrix}\; x\right]\right\}^2 = e^{x}{}_1F_2\left[\begin{matrix} a; \\ a+\frac{1}{2},\; 2a; \end{matrix}\; \frac{x^2}{4}\right]. \tag{1.9} $$

Rathie [8] proved the Preece's identity (1.8) very shortly by using the following product formula due to Bailey [1]:

$$ {}_0F_1\left[\begin{matrix} —; \\ \varrho; \end{matrix}\; x\right] \cdot {}_0F_1\left[\begin{matrix} —; \\ \sigma; \end{matrix}\; x\right] = {}_2F_3\left[\begin{matrix} \frac{1}{2}(\varrho+\sigma),\; \frac{1}{2}(\varrho+\sigma-1); \\ \varrho,\; \sigma,\; \varrho+\sigma-1; \end{matrix}\; 4x\right] \tag{1.10} $$

and the Kummer's second transformation formula (1.2). Rathie [8] also presented the following two more results closely related to (1.8):

$$
{}_1F_1\left[\begin{matrix} a; \\ 2a; \end{matrix}\, x\right] {}_1F_1\left[\begin{matrix} a; \\ 2a+1; \end{matrix}\, x\right]
$$

$$
= e^x \left\{ {}_1F_2\left[\begin{matrix} a; \\ 2a,\ a+\frac{1}{2}; \end{matrix}\, \frac{x^2}{4}\right] - \frac{x}{2(2a+1)}\, {}_1F_2\left[\begin{matrix} a+1; \\ 2a+1,\ a+\frac{3}{2}; \end{matrix}\, \frac{x^2}{4}\right] \right\} \quad (1.11)
$$

and

$$
{}_1F_1\left[\begin{matrix} a; \\ 2a; \end{matrix}\, x\right] {}_1F_1\left[\begin{matrix} a; \\ 2a-1; \end{matrix}\, x\right]
$$

$$
= e^x \left\{ {}_1F_2\left[\begin{matrix} a; \\ 2a-1,\ a+\frac{1}{2}; \end{matrix}\, \frac{x^2}{4}\right] + \frac{x}{2(2a-1)}\, {}_1F_2\left[\begin{matrix} a; \\ 2a,\ a+\frac{1}{2}; \end{matrix}\, \frac{x^2}{4}\right] \right\}. \quad (1.12)
$$

Here, in this paper, we aim to provide explicit expressions of

$$
\left\{ {}_1F_1\left[\begin{matrix} a; \\ 2a+i; \end{matrix}\, x\right] \right\}^2, \quad (1.13)
$$

$$
\left\{ {}_1F_1\left[\begin{matrix} a; \\ 2a-i; \end{matrix}\, x\right] \right\}^2 \quad (1.14)
$$

and

$$
{}_1F_1\left[\begin{matrix} a; \\ 2a+i; \end{matrix}\, x\right] {}_1F_1\left[\begin{matrix} a; \\ 2a-i; \end{matrix}\, x\right] \quad (1.15)
$$

in the most general form for any non-negative integer i, by using the same method as in Rathie [8]. For our purpose, we need to recall two general results in [6], which are written here in the following slightly modified forms:

$$
e^{-x/2}\, {}_1F_1\left[\begin{matrix} a; \\ 2a+i; \end{matrix}\, x\right]
$$

$$
= \sum_{m=0}^{i} \frac{(-i)_m (2a-1)_m}{(2a+i)_m \left(a-\frac{1}{2}\right)_m m!} \frac{x^m}{4^m}\, {}_0F_1\left[\begin{matrix} -; \\ a+m+\frac{1}{2}; \end{matrix}\, \frac{x^2}{16}\right] \quad (1.16)
$$

and

$$e^{-x/2}{}_1F_1\left[\begin{matrix} a; \\ 2a-i; \end{matrix}\, x\right]$$

$$= \sum_{m=0}^{i} \frac{(-1)^m(-i)_m(2a-2i-1)_m}{(2a-i)_m\left(a-i-\frac{1}{2}\right)_m m!}\frac{x^m}{4^m}\,{}_0F_1\left[\begin{matrix} \text{---}; \\ a+m-i+\frac{1}{2}; \end{matrix}\, \frac{x^2}{16}\right] \tag{1.17}$$

for each $i \in \mathbb{N}_0$.

2. Main Results

Here we present two general formulas involving the square of the generalized hypergeometric functions and one companion result asserted in the following theorem.

Theorem 1. *Each of the following formulas holds true for* $i \in \mathbb{N}_0$:

$$\left\{{}_1F_1\left[\begin{matrix} a; \\ 2a+i; \end{matrix}\, x\right]\right\}^2$$

$$= e^x \sum_{m=0}^{i}\sum_{n=0}^{i} \frac{(-i)_m(-i)_n(2a-1)_m(2a-1)_n}{\left(a-\frac{1}{2}\right)_m\left(a-\frac{1}{2}\right)_n(2a+i)_m(2a+i)_n}\frac{x^{m+n}}{m!\,n!2^{2m+2n}}$$

$$\times\,{}_2F_3\left[\begin{matrix} \frac{1}{2}(2a+m+n+1),\ \frac{1}{2}(2a+m+n); \\ a+m+\frac{1}{2},\ a+n+\frac{1}{2},\ 2a+m+n; \end{matrix}\, \frac{x^2}{4}\right], \tag{2.1}$$

$$\left\{{}_1F_1\left[\begin{matrix} a; \\ 2a-i; \end{matrix}\, x\right]\right\}^2$$

$$= e^x \times \sum_{m=0}^{i}\sum_{n=0}^{i} \frac{(-1)^{m+n}(-i)_m(-i)_n(2a-2i-1)_m(2a-2i-1)_n}{\left(a-i-\frac{1}{2}\right)_m\left(a-i-\frac{1}{2}\right)_n(2a-i)_m(2a-i)_n}\frac{x^{m+n}}{m!\,n!2^{2m+2n}}$$

$$\times\,{}_2F_3\left[\begin{matrix} \frac{1}{2}(2a+m+n-2i+1),\ \frac{1}{2}(2a+m+n-2i); \\ a+m-i+\frac{1}{2},\ a+n-i+\frac{1}{2},\ 2a+m+n-2i; \end{matrix}\, \frac{x^2}{4}\right] \tag{2.2}$$

and

$$
\begin{aligned}
&{}_1F_1\left[\begin{matrix} a; \\ 2a+i; \end{matrix}\; x\right] {}_1F_1\left[\begin{matrix} a; \\ 2a-i; \end{matrix}\; x\right] \\
&= e^x \sum_{m=0}^{i}\sum_{n=0}^{i} \frac{(-1)^n (-i)_m (-i)_n (2a-1)_m (2a-2i-1)_n}{\left(a-\frac{1}{2}\right)_m \left(a-i-\frac{1}{2}\right)_n (2a+i)_m (2a-i)_n} \frac{x^{m+n}}{4^{m+n}\, m!\, n!} \\
&\quad \times {}_2F_3\left[\begin{matrix} \frac{1}{2}(2a+m+n-i+1),\ \frac{1}{2}(2a+m+n-i); \\ a+m+\frac{1}{2},\ a+n-i+\frac{1}{2},\ 2a+m+n-i; \end{matrix}\; \frac{x^2}{4}\right]. \qquad (2.3)
\end{aligned}
$$

Proof. For (2.1), it is sufficient to show the following statement:

$$
\begin{aligned}
&e^{-x}\left\{{}_1F_1\left[\begin{matrix} a; \\ 2a+i; \end{matrix}\; x\right]\right\}^2 \\
&= \sum_{m=0}^{i}\sum_{n=0}^{i} \frac{(-i)_m (-i)_n (2a-1)_m (2a-1)_n}{\left(a-\frac{1}{2}\right)_m \left(a-\frac{1}{2}\right)_n (2a+i)_m (2a+i)_n} \frac{x^{m+n}}{m!\, n! 2^{2m+2n}} \\
&\quad \times {}_2F_3\left[\begin{matrix} \frac{1}{2}(2a+m+n+1),\ \frac{1}{2}(2a+m+n); \\ a+m+\frac{1}{2},\ a+n+\frac{1}{2},\ 2a+m+n; \end{matrix}\; \frac{x^2}{4}\right] \quad (i \in \mathbb{N}_0). \qquad (2.4)
\end{aligned}
$$

Now, let $\mathcal{L}$ be the left-hand side of (2.4) and be separated in the following form:

$$
\mathcal{L} = \left\{e^{-x/2}{}_1F_1\left[\begin{matrix} a; \\ 2a+i; \end{matrix}\; x\right]\right\}\left\{e^{-x/2}{}_1F_1\left[\begin{matrix} a; \\ 2a+i; \end{matrix}\; x\right]\right\}. \qquad (2.5)
$$

Applying (1.16) to each term of (2.5), we obtain

$$
\begin{aligned}
\mathcal{L} &= \sum_{m=0}^{i}\sum_{n=0}^{i} \frac{(-i)_m (-i)_n (2a-1)_m (2a-1)_n}{\left(a-\frac{1}{2}\right)_m \left(a-\frac{1}{2}\right)_n (2a+i)_m (2a+i)_n} \frac{x^{m+n}}{m!\, n! 2^{2m+2n}} \\
&\quad \times {}_0F_1\left[\begin{matrix} —; \\ a+m+\frac{1}{2}; \end{matrix}\; \frac{x^2}{16}\right] {}_0F_1\left[\begin{matrix} —; \\ a+n+\frac{1}{2}; \end{matrix}\; \frac{x^2}{16}\right]. \qquad (2.6)
\end{aligned}
$$

Using (1.10) for the product of two ${}_0F_1$ in the right-hand side of (2.6), it is easy to see that the $\mathcal{L}$ is equal to the right-hand side of (2.1). This completes the proof of (2.1).

The same argument as in the proof of (2.1) (here using (1.17) instead of (1.16)) will establish the identities (2.2) and (2.3). So the details of their proofs are omitted. □

3. Special Cases and Concluding Remark

Here we consider some special cases of our main results in Theorem 1 and give a remark for a possible further research related to the present investigation.

(1) In (2.1), (2.2), or (2.3), if we take $i = 0,$ then we immediately recover the Preece's identity (1.9).

(2) Taking $i = 1$ in (2.1) yields the following result:

$$\left\{{}_1F_1\left[\begin{matrix} a; \\ 2a+1; \end{matrix}\ x\right]\right\}^2$$
$$= e^x\left\{{}_1F_2\left[\begin{matrix} a; \\ a+\frac{1}{2},\ 2a; \end{matrix}\ \frac{x^2}{4}\right] - \frac{x}{2a+1}\,{}_1F_2\left[\begin{matrix} a+1; \\ a+\frac{3}{2},\ 2a+1; \end{matrix}\ \frac{x^2}{4}\right]\right.$$
$$\left. + \frac{x^2}{4(2a+1)^2}\,{}_1F_2\left[\begin{matrix} a+1; \\ a+\frac{3}{2},\ 2a+2; \end{matrix}\ \frac{x^2}{4}\right]\right\}. \tag{3.1}$$

(3) Taking $i = 1$ in (2.2) yields the following result:

$$\left\{{}_1F_1\left[\begin{matrix} a; \\ 2a-1; \end{matrix}\ x\right]\right\}^2$$
$$= e^x\left\{{}_1F_2\left[\begin{matrix} a-1; \\ a-\frac{1}{2},\ 2a-2; \end{matrix}\ \frac{x^2}{4}\right] + \frac{x}{2a-1}\,{}_1F_2\left[\begin{matrix} a; \\ a+\frac{1}{2},\ 2a-1; \end{matrix}\ \frac{x^2}{4}\right]\right.$$
$$\left. + \frac{x^2}{4(2a-1)^2}\,{}_1F_2\left[\begin{matrix} a; \\ a+\frac{1}{2},\ 2a; \end{matrix}\ \frac{x^2}{4}\right]\right\}. \tag{3.2}$$

(4) Taking $i = 1$ in (2.3), we get the following result:

$$
{}_1F_1\left[\begin{matrix} a; \\ 2a+1; \end{matrix}\ x\right]{}_1F_1\left[\begin{matrix} a; \\ 2a-1; \end{matrix}\ x\right]
$$

$$
= e^x\left\{{}_1F_2\left[\begin{matrix} a; \\ a+\frac{1}{2},\ 2a-1; \end{matrix}\ \frac{x^2}{4}\right] + \frac{x}{2(2a-1)}\,{}_1F_2\left[\begin{matrix} a; \\ a+\frac{1}{2},\ 2a; \end{matrix}\ \frac{x^2}{4}\right]\right.
$$

$$
- \frac{x}{2(2a+1)}\,{}_2F_3\left[\begin{matrix} a,\ a+\frac{1}{2}; \\ a-\frac{1}{2},\ a+\frac{3}{2},\ 2a; \end{matrix}\ \frac{x^2}{4}\right]
$$

$$
\left. - \frac{x^2}{4(2a-1)(2a+1)}\,{}_1F_2\left[\begin{matrix} a+1; \\ a+\frac{3}{2},\ 2a+1; \end{matrix}\ \frac{x^2}{4}\right]\right\}. \tag{3.3}
$$

We conclude this paper by noting that, using these general formulas presented here, we may obtain two further general transformation formulas for the double hypergeometric functions such as (for example) Srivastava and Panda's double hypergeometric functions (see, e.g., [11, p. 27]).

References

[1] W. N. Bailey, Product of generalized hypergeometric series, Proc. London Math. Soc. (Ser. 2) 28 (1928), 242-254.

[2] W. N. Bailey, Generalized Hypergeometric Series, Cambridge University Press, Cambridge, 1935, Reprinted by Stechert Hafner, New York, 1964.

[3] C. F. Gauss, Disquisitiones Generaies Circa Seriem Infinimum, Gottengen: Vol. II, 437-445; Vol. III, pp. 123-163, 207-229 and 446-460.

[4] E. E. Kummer, Uber die hypergeometriche Reihe, J. Reine Angew. Math. 15 (1836), pp. 39-83 and 127-172.

[5] C. T. Preece, The product of two generalized hypergeometric function, Proc. London Math. Soc. (Ser. 2) 22 (1924), 370-380.

[6] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, Integrals and Series, Vol. 3, More Special Functions, Gordon and Breach Science Publishers, New York, 1990.

[7] E. D. Rainville, Special Functions, Macmillan Company, New York, 1960; Reprinted by Chelsea Publishing Company, Bronx, New York, 1971.

[8] A. K. Rathie, A short proof of Preece's identity and other contiguous results, Rev. Mat. Estat. 15 (1997), 207-210.

[9] A. K. Rathie and J. Choi, Another proof of Kummer's second theorem, Commun. Korean Math. Soc. 13 (1998), 933-936.

[10] H. M. Srivastava and J. Choi, Zeta and q-Zeta Functions and Associated Series and Integrals, Elsevier Science Publishers, Amsterdam, London and New York, 2012.

[11] H. M. Srivastava and P. W. Karlsson, Multiple Gaussian Hypergeometric Series, Ellis Horwood Series: Mathematics and its Applications, Ellis Horwood Ltd., Chichester; Halsted Press, John Wiley & Sons, Inc., New York, 1985.